Optimal investment with high-watermark performance fee

Mihai Sîrbu, University of Texas at Austin

based on joint work with

Karel Janeček
RSJ Algorithmic Trading and Charles University

SIAM Conference on Financial Mathematics & Engineering,
San Francisco, November 19-20, 2010

Outline

Objective

- build and analyze a model of optimal investment and consumption where the investment opportunity is represented by a hedge-fund using the "two-and-twenty rule"
- analyze the impact of the high-watermark fee on the investor

Previous work on hedge-funds and high-watermarks

All existing work analyzes the impact/incentive of the high-watermark fees on fund managers

- extensive finance literature
  - Goetzmann, Ingersoll and Ross, Journal of Finance 2003
  - Panagea and Westerfield, Journal of Finance 2009
  - Agarwal, Daniel and Naik Journal of Finance, forthcoming
  - Aragon and Qian, preprint 2007
- recently studied in mathematical finance
  - Guasoni and Obloj, preprint 2009

A model of profits from dynamically investing in a hedge-fund

- the investor chooses to hold θ_t in the fund at time t
- the value of the fund F_t is given exogenously
- denote by P_t the accumulated profit/losses up to time t

Evolution of the profit

- without high-watermark fee

$$dP_t = \theta_t \frac{dF_t}{F_t}, \quad P_0 = 0$$

- with high-watermark proportional fee $\lambda > 0$

$$\begin{cases} dP_t = \theta_t \frac{dF_t}{F_t} - \lambda dP_t^*, \quad P_0 = 0 \\ P_t^* = \max_{0 \le s \le t} P_s \end{cases}$$

High-watermark of the investor

$$P_t^* = \max_{0 \le s \le t} P_s.$$

Observation: can be also interpreted as taxes on gains, paid right when gains are realized (pointed out by Paolo Guasoni)

Path-wise solutions

(same as Guasoni and Obloj)

Denote by I_t the paper profits from investing in the fund

$$I_t = \int_0^t \theta_u \frac{dF_u}{F_u}$$

Then

$$P_t = I_t - \frac{\lambda}{\lambda + 1} \max_{0 \le s \le t} I_s$$

The high-watermark of the investor is

$$P_t^* = \frac{1}{\lambda + 1} \max_{0 \le s \le t} I_s$$

Observations:

- the fee λ can exceed 100% and the investor can still make a profit
- the high-watemark is measured before the fee is paid

Connection to the Skorohod map (Part of work in progress with Gerard Brunick)

Denote by $Y = P^* - P$ the distance from paying fees. Then Y satisfies the equation:

$$\begin{cases} dY_t = -\theta_t \frac{dF_t}{F_t} + (1+\lambda) dP^*_t \\ Y_0 = 0, \end{cases}$$

where $Y \geq 0$ and

$$\int_0^t \mathbb{I}_{\{Y_s \neq 0\}} dP^*_s = 0, \quad (\forall)\ t \geq 0.$$

Skorohod map

$$I_{\cdot} = \int_0^{\cdot} \theta_u \frac{dF_u}{F_u} \rightarrow (Y, P^*) \approx (P, P^*).$$

Remark: Y will be chosen as state in more general models.

The model of investment and consumption

An investor with initial capital $x > 0$ chooses to

- have θ_t in the fund at time t
- consume at a rate γ_t
- finance from borrowing/investing in the money market at zero rate

Denote by $C_t = \int_0^t \gamma_s ds$ the accumulated consumption. Since the money market pays zero interest, then

$$X_t = x + P_t - C_t \leftrightarrow P_t = (X_t + C_t) - x$$

Therefore, the fees (high-watermark) is computed tracking the wealth and accumulated consumption

$$P_t^* = \max_{0 \leq s \leq t} \left\{ X_s + \int_0^s \gamma_u du \right\} - x$$

Can think that the investor leaves all her wealth (including the money market) with the investor manager.

Evolution equation for the wealth

The evolution of the wealth is

$$\begin{cases} dX_t = \theta_t \frac{dF_t}{F_t} - \gamma_t dt - \lambda dP_t^*, \quad X_0 = x \\ P_t^* = \max_{0 \le s \le t} \left\{ X_s + \int_0^s \gamma_u du \right\} - x \end{cases}$$

- consumption is a part of the running-max, as opposed to the literature on draw-dawn constraints
  - Grossman and Zhou
  - Cvitanic and Karatzas
  - Elie and Touzi
  - Roche
- we still have a similar path-wise representation for the wealth in terms of the "paper profit" I_t and the accumulated consumption

Optimal investment and consumption

Admissible strategies

$$\mathscr{A}(x) = \{(\theta, \gamma) : X > 0\}.$$

Can represent investment and consumption strategies in terms of proportions

$$c = \gamma / X, \quad \pi = \theta.$$

Obervation:

- no closed form path-wise solutions for X in terms of (π, c) (unless $c = 0$)

Optimal investment and consumption:cont'd

Maximize discounted utility from consumption on infinite horizon

$$\mathscr{A}(x) \ni (\theta, \gamma) \to \operatorname{argmax} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt\right].$$

Where $U : (0, \infty) \to \mathbb{R}$ is the CRRA utility

$$U(\gamma) = \frac{\gamma^{1-p}}{1-p}, \quad p > 0.$$

Finally, choose a geometric Brownian-Motion model for the fund share price

$$\frac{dF_t}{F_t} = \alpha dt + \sigma dW_t.$$

Dynamic programming: state processes

Fees are paid when $P = P^*$. This can be translated as $X + C = (X + C)^*$ or as

$$X = (X + C)^* - C.$$

Denote by

$$N \triangleq (X + C)^* - C.$$

The (state) process (X, N) is a two-dimensional controlled diffusion $0 < X \leq N$ with reflection on $\{X = N\}$.
The evolution of the state (X, N) is given by

$$\begin{cases} dX_t = \big(\theta_t \alpha - \gamma_t\big) dt + \theta_t \sigma dW_t - \lambda dP^*_t, \; X_0 = x \\ dN_t = -\gamma_t dt + dP^*_t, \quad N_0 = x. \end{cases}$$

Recall we have path-wise solutions in terms of (θ, γ).

Dynamic Programming: Objective

- we are interested to solve the problem using dynamic programing. We are only interested in the initial condition (x, n) for $x = n$ but we actually solve the problem for all $0 < x \leq n$. This amounts to setting an initial high-watemark of the investor which is larger than the initial wealth.
- expect to find the two-dimensional value function $v(x, n)$ as a solution of the HJB, and find the (feed-back) optimal controls.

Dynamic programming equation

Use Itô and write formally the HJB

$$\sup_{\gamma \geq 0, \theta} \left\{ -\beta v + U(\gamma) + (\alpha\theta - \gamma) v_x + \frac{1}{2}\sigma^2\theta^2 v_{xx} - \gamma v_n \right\} = 0$$

for $0 < x < n$ and the boundary condition

$$-\lambda v_x(x,x) + v_n(x,x) = 0.$$

(Formal) optimal controls

$$\hat{\theta}(x,n) = -\frac{\alpha}{\sigma^2} \frac{v_x(x,n)}{v_{xx}(x,n)}$$

$$\hat{\gamma}(x,n) = I(v_x(x,n) + v_n(x,n))$$

HJB cont'd

Denote by $\tilde{U}(y) = \frac{p}{1-p} y^{\frac{p-1}{p}}, \quad y > 0$ the dual function of the utility. The HJB becomes

$$-\beta v + \tilde{U}(v_x + v_n) - \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{v_x^2}{v_{xx}} = 0, \quad 0 < x < n$$

plus the boundary condition

$$-\lambda v_x(x,x) + v_n(x,x) = 0.$$

Observation:

- if there were no v_n term in the HJB, we could solve it closed-form as in Roche or Elie-Touzi using the (dual) change of variable $y = v_x(x,n)$
- no closed-from solutions in our case (even for power utility)

Reduction to one-dimension

Since we are using power utility

$$U(x) = \frac{x^{1-p}}{1-p}, \quad p > 0$$

we can reduce to one-dimension

$$v(x, n) = x^{1-p} v(1, \frac{n}{x})$$

and

$$v(x, n) = n^{1-p} v(\frac{x}{n}, 1)$$

- first is nicer economically (since for $\lambda = 0$ we get a constant function $v(1, \frac{n}{x})$)
- the second gives a nicer ODE (works very well if there is a closed form solution, see Roche)

There is no closed form solution, so we can choose either one-dimensional reduction.

Reduction to one-dimension cont'd

We decide to denote $z = \frac{n}{x} \geq 1$ and

$$v(x, n) = x^{1-p} u(z).$$

Use

$$\begin{aligned} v_n(x, n) &= u'(z) \cdot x^{-p}, \\ v_x(x, n) &= \Big((1-p)u(z) - zu'(z) \Big) \cdot x^{-p}, \\ v_{xx}(x, n) &= \Big(-p(1-p)u(z) + 2pzu'(z) + z^2 u''(z) \Big) \cdot x^{-1-p}, \end{aligned}$$

to get the reduced HJB

$$-\beta u + \tilde{U}\big((1-p)u - (z-1)u'\big) - \frac{1}{2}\frac{\alpha^2}{\sigma^2}\frac{\big((1-p)u - zu'\big)^2}{-p(1-p)u + 2pzu' + z^2 u''} = 0$$

for $z > 1$ with boundary condition

$$-\lambda(1-p)u(1) + (1+\lambda)u'(1) = 0$$

(Formal) optimal proportions

$$\hat{\pi}(z) = \frac{\alpha}{p\sigma^2} \cdot \frac{(1-p)u - zu'}{(1-p)u - 2zu' - \frac{1}{p}z^2u''},$$

$$\hat{c}(z) = \frac{(v_x + v_n)^{-\frac{1}{p}}}{x} = \left((1-p)u - (z-1)u'\right)^{-\frac{1}{p}}$$

Optimal amounts (controls)

$$\hat{\theta}(x,n) = x\hat{\pi}(z), \quad \hat{\gamma}(x,n) = x\hat{c}(z)$$

Objective: solve the HJB analytically and then do verification

Solution of the HJB for $\lambda = 0$

This is the classical Merton problem. The optimal investment proportion is given by

$$\pi_0 \triangleq \frac{\alpha}{p\sigma^2},$$

while the value function equals

$$v_0(x, n) = \frac{1}{1-p}\, c_0^{-p} x^{1-p}, \quad 0 < x \leq n,$$

where

$$c_0 \triangleq \frac{\beta}{p} - \frac{1}{2}\,\frac{1-p}{p^2} \cdot \frac{\alpha^2}{\sigma^2}$$

is the optimal consumption proportion. It follows that the one-dimensional value function is constant

$$u_0(z) = \frac{1}{1-p}\, c_0^{-p}, \qquad z \geq 1.$$

Solution of the HJB for $\lambda > 0$

If $\lambda > 0$ we expect that (additional boundary condition)

$$\lim_{z \to \infty} u(z) = u_0.$$

(For very large high-watermark, the investor gets almost the Merton expected utility)

Existence of a smooth solution

Theorem 1 The HJB has a smooth solution.

Idea of solving the HJB:

- find a viscosity solution using an adaptation of Perron's method. Consider infimum of concave supersolutions that satisfy the boundary condition. Obtain as a result a concave viscosity solution. The subsolution part is more delicate. Have to treat carefully the boundary condition.

Proof of existence: cont'd

- show that the viscosity solution is C^2 (actually more). Concavity, together with the subsolution property implies C^1 (no kinks). Go back into the ODE and formally rewrite it as

$$u'' = f(z, u(z), u'(z)) \triangleq g(z).$$

  Compare locally the viscosity solution u with the classical solution of a similar equation

$$w'' = g(z)$$

  with the same boundary conditions, whenever u, u' are such that g is continuous. The difficulty is to show that u, u' always satisfy this requirement.

Avoid defining the value function and proving the Dynamic Programming Principle.

Verification, Part I

Theorem 2 The closed loop equation

$$\begin{cases} dX_t = \hat{\theta}(X_t, N_t)\frac{dF_t}{F_t} - \hat{\gamma}(X_t, N_t)dt - \lambda(dN_t + \gamma_t dt), \quad X_0 = x \\ N_t = \max_{0\le s\le t}\left\{X_s + \int_0^s \hat{\gamma}(X_u, N_u)du\right\} - \int_0^t \hat{\gamma}(X_u, N_u)du \end{cases}$$

has a unique strong solution $0 < \hat{X} \le \hat{N}$.

Ideea of proof:

- use the path-wise representation

$$(Y, L) \to (\hat{\theta}(Y, L), \hat{\gamma}(Y, L)) \to (X, N)$$

  together with the Itô-Picard theory to obtain a unique global solution $X \le N$.

- use the fact that the optimal proportion $\hat{\pi}$ and $\hat{c}$ are bounded to compare $\hat{X}$ to an exponential martingale and conclude

$$\hat{X} > 0$$

Verification, Part II

Theorem 3 The controls $\hat{\theta}(\hat{X}_t, \hat{N}_t)$ and $\hat{\gamma}(\hat{X}_t, \hat{N}_t)$ are optimal.

Idea of proof:

- use Itô together with the HJB to conclude that

$$e^{-\beta t} V(X_t, N_t) + \int_0^t e^{-\beta s} U(\gamma_s) ds, \quad 0 \leq t < \infty,$$

  is a local supermartingale in general and a local martingale for the candidate optimal controls (the obvious part)
- uniform integrability. Has to be done separately for $p < 1$ and $p > 1$ (the harder part, requires again the use of $\hat{\pi}$ and $\hat{c}$ bounded, and comparison to an exponential martingale).

The impact of fees

Everything else being equal, the fees have the effect of

- reducing rate of return
- reducing initial wealth

Certainty equivalent return

We consider two investors having the same initial wealth, risk-aversion, who invest in two funds with the same volatility

- one invests in a fund with return α, and pays fees $\lambda > 0$. The initial high-watermark is $n = xz \geq x$
- the other invests in a fund with return $\tilde{\alpha}$ but pays no fees

Equate the expected utilities:

$$u_0\big(\tilde{\alpha}(z), \cdot\big) = u_\lambda(\alpha, z).$$

Can be solved as

$$\tilde{\alpha}^2(z) = 2\sigma^2 \frac{p^2}{1-p}\left(\frac{\beta}{p} - \big((1-p)u_\lambda(z)\big)^{-\frac{1}{p}}\right), \quad z \geq 1.$$

The relative size of the certainty equivalent excess return is therefore

$$\frac{\tilde{\alpha}(z)}{\alpha} = \frac{\sqrt{2}\sigma p}{\alpha}\left(\frac{\frac{\beta}{p} - \big((1-p)u_\lambda(z)\big)^{-\frac{1}{p}}}{1-p}\right)^{\frac{1}{2}}, \quad z \geq 1.$$

Certainty equivalent initial wealth

We consider two investors having the same risk-aversion, who invest in the same fund

- one has initial wealth x, initial high-watermark $n = xz \geq x$ and pays fees $\lambda > 0$
- the other has initial wealth $\tilde{x}$ but pays no fees

Equate the expected utilities:

$$\tilde{x}(z)^{1-p} u_0(\cdot) = v_0(\tilde{x}(z), \cdot) = v_\lambda(x, n) = x^{1-p} u_\lambda(z)$$

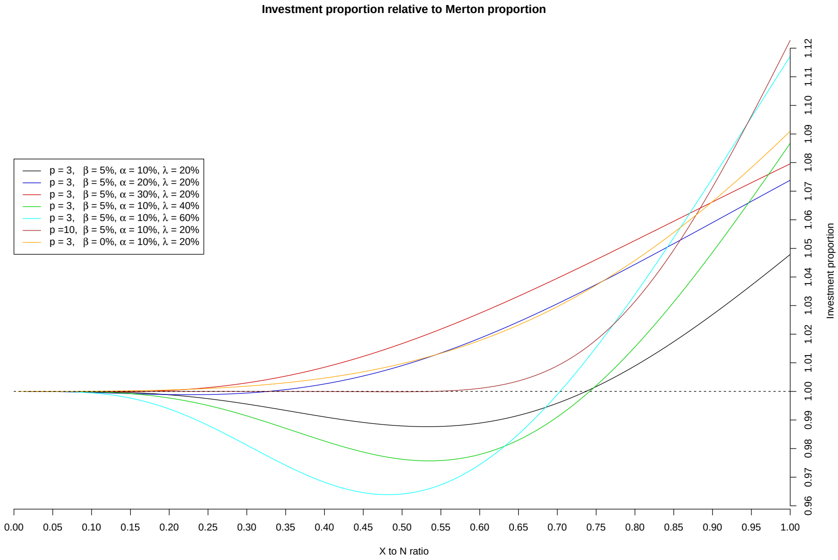
all other parameters being equal. Can be solved as

$$\tilde{x}(z) = x \cdot \left(\frac{u_\lambda(z)}{u_0} \right)^{\frac{1}{1-p}} = x \cdot \left((1-p) c_0^p \, u_\lambda(z) \right)^{\frac{1}{1-p}}, \quad z \geq 1.$$

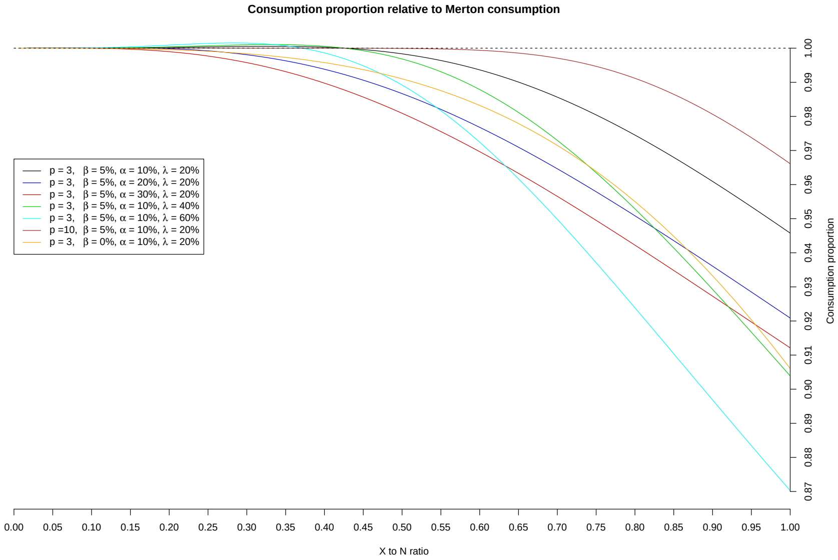
The quantity

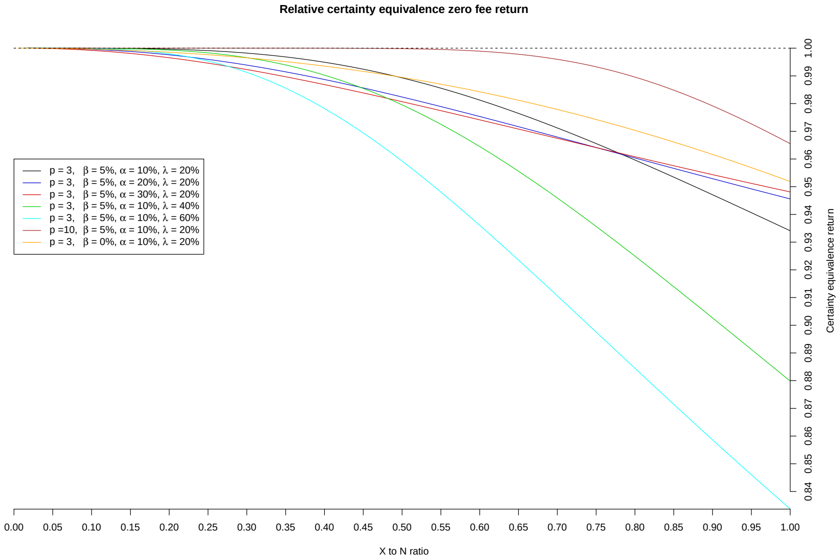
$$\frac{\tilde{x}(z)}{x} = \left(\frac{u_\lambda(z)}{u_0} \right)^{\frac{1}{1-p}} = \left((1-p) c_0^p \, u_\lambda(z) \right)^{\frac{1}{1-p}}, \quad z \geq 1,$$

is the relative certainty equivalent wealth.

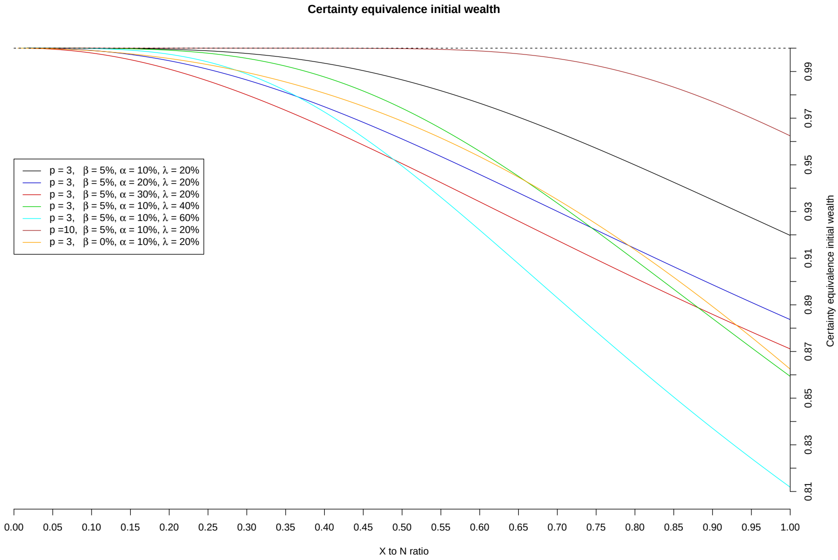


Investment proportion relative to Merton proportion
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
X to N ratio
0.96 0.97 0.98 0.99 1.00 1.01 1.02 1.03 1.04 1.05 1.06 1.07 1.08 1.09 1.10 1.11 1.12
Investment proportion


Consumption proportion relative to Merton consumption
p = 3,   β = 5%, α = 10%, λ = 20%
p = 3,   β = 5%, α = 20%, λ = 20%
p = 3,   β = 5%, α = 30%, λ = 20%
p = 3,   β = 5%, α = 10%, λ = 40%
p = 3,   β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3,   β = 0%, α = 10%, λ = 20%
0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45 0.50 0.55 0.60 0.65 0.70 0.75 0.80 0.85 0.90 0.95 1.00
X to N ratio
0.87 0.88 0.89 0.90 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98 0.99 1.00
Consumption proportion



Relative certainty equivalence zero fee return
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
X to N ratio
Certainty equivalence return
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
0.84
0.85
0.86
0.87
0.88
0.89
0.90
0.91
0.92
0.93
0.94
0.95
0.96
0.97
0.98
0.99
1.00



Certainty equivalence initial wealth
p = 3, β = 5%, α = 10%, λ = 20%
p = 3, β = 5%, α = 20%, λ = 20%
p = 3, β = 5%, α = 30%, λ = 20%
p = 3, β = 5%, α = 10%, λ = 40%
p = 3, β = 5%, α = 10%, λ = 60%
p =10, β = 5%, α = 10%, λ = 20%
p = 3, β = 0%, α = 10%, λ = 20%
0.99
0.97
0.95
0.93
0.91
0.89
0.87
0.85
0.83
0.81
Certainty equivalence initial wealth
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
X to N ratio

Conclusions

Point of view of Finance:

- model optimal investment with high-watermark fees from the point of view of the investor
- analyze the impact of the fees

Point of Mathematics:

- an example of controlling a two-dimensional reflected diffusion
- solve the problem using direct dynamic programming: first find a smooth solution of the HJB and then do verification

"Meta Conclusion":

- whenever one can prove enough regularity for the viscosity solution to do verification, the viscosity solution can/should be constructed analytically, using Perron's method, and avoiding DPP altogether

Work in progress and future work

with Gerard Brunick and Karel Janeček

- presence of (multiple and correlated) traded stocks, interest rates and hurdles: can still be modeled as a two-dimensional diffusion problem using X and $Y = P - P^*$ as state processes (reduced to one-dimension by scaling)
- analytic approximations when λ is small
- more than one fund: genuinely multi-dimensional problem with reflection
- stochastic volatility, jumps, etc

Where does it all go?

Investor

- can either invest in a number of assets $(S_1, \ldots, S_n)$ with transaction costs
- invest in the hedge-fund F paying profit fees.

The hedge-fund

- can invest in the assets with lower (even zero for mathematical reasons) transaction costs, and produce the fund process F.

For certain choices of F (time-dependent combinations of the stocks and money market), one can compare the utility of the investor in the two situations: this should the existence of hedge-funds (from the point of view of the investor).

Actually, the whole situation should be modeled as a game between the investor and the hedge fund.