The low risk free rate is not too low*

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Abstract

One of the puzzles of the modern economics theory is related to the size of the risk free rate. Standard results suggest that in order to explain the low observed risk free rate in conjunction with consumption growth data, one needs to assume very low risk aversion of the representative agent. Low risk aversion in turn contradicts the observed equity premium.

In this paper I argue that the observed risk free rate is consistent with both observed consumption data and plausible equity premium. The remaining piece of puzzle is to adapt sufficiently high but realistic risk aversion coefficient.

1 Realistic risk aversion coefficient

It appears plausible that the usage of CRRA utility function could be a reasonable approximation of the real-world behavior. I assume the power (CRRA) utility function of the form:

$$U_p(c_t) = \begin{cases} \frac{c_t^{1-p}-1}{1-p}, & p > 0,\ p \neq 1 \\ \log c_t, & p = 1. \end{cases} \tag{1}$$

The assumption of power utility functions can be also understood in the following sense: While an individual agent may not possess exactly the power utility function, we are trying to use the constant relative risk-aversion coefficient, which would "best" approximate the agent's risk-taking behavior. In other words, we attempt to answer the problem of what risk aversion coefficient would the agent choose, if one "forces" her to use the power utility function for portfolio and consumption choice decisions.

In this sense it is important to note that the considered risk aversion coefficients are designed to match the realistic aversion to risk for relatively small risks and low expected return, with not too skewed distribution of the results. Taking this kind of risk corresponds to investing in financial markets.

In (Janeček 2002) I argue that a realistic risk aversion coefficient for an investor on financial markets is probably higher than $p = 30$. Sufficiently high risk aversion coefficient enables to explain the *equity premium puzzle* (see (Mehra and Prescott 1985)), and other properties of economics data. See also (Kandel and Stambaugh 1990) for similar arguments advocating the use of high risk aversion coefficients.

I further argue in (Janeček 2002) that many households may exhibit risk aversion coefficients of 200 and more. All agents, including the very risk averse agents, influence the consumption data, and consequently also the size of risk free rate. Thus, it is plausible that, for the purposes of risk free rate analysis, the risk aversion of a representative agent may easily exceed $p = 100$.

2 Low risk free rate as a result of partial equilibrium

We assume that the representative agent maximizes the value function

$$v(W_0) = \sup_{C,\Pi\in\mathcal{A}(W_0)} \mathrm{E} \int_0^\infty \mathrm{e}^{-\beta t}\, U_p(c_t)\, dt \tag{2}$$

for admissible controls $C = \{c_t, t \geq 0\}$ (the consumption process) and $\Pi = \{\pi_t, t \geq 0\}$ (the process of proportion of wealth invested in stock). U_p denotes the power utility function, W_0 is the initial investment wealth, and β is the subjective discount rate,.

For simplicity of exposition, we further assume a geometric Brownian motion for the consumption process

$$dc_t = c_t\big(\alpha\, dt + \sigma\, dB_t\big).$$

This setup is a special case of general equilibrium model in (Karatzas, Lakner, Lehoczky, and Shreve 1991). For the equilibrium risk free rate r we obtain, see (Karatzas, Lakner, Lehoczky, and Shreve 1991)[p. 264],

$$r = \beta + \alpha \cdot p - \frac{1}{2}\sigma^2 \cdot p(p+1). \tag{3}$$

Equation (3) gives the *shadow price* of individual consumer's risk free interest rate. If the market risk free rate was higher the consumer would adjust her position by consuming less and investing more in a risk free asset. If the risk free rate was lower the consumer would (in a frictionless market) borrow at the risk free rate and consume more.

Formula (3) shows that consumption growth is high when interest rates are high. Since higher coefficient of risk aversion p makes interest rates more sensitive to consumption growth, an argument is often made that higher risk aversion implies higher interest rate, and that a high risk aversion coefficient cannot explain the low risk free rate observed in the economy.[1] People would often ignore the *precautionary savings* effect expressed by $\frac{1}{2}\sigma^2 \cdot p(p+1)$ since the observed volatility of consumption is low (say in the order of 3%). My point is that the size of risk free rate is a quadratic function of p. For sufficiently high p, the effect of precautionary savings starts to dominate the effect of consumption growth. We will see that the interest rate puzzle is satisfactory explained when considering sufficiently high (and plausible) risk aversion coefficients, and when confronting the model with real world consumption growth and volatility of consumption data.[2]

In this context it is interesting to note the results in (Weil 1989). The author attempts to resolve the equity premium puzzle and risk free rate puzzle with a class of more general Kreps-Porteus non-expected utility framework with independent parametrization of attitudes toward risk and attitudes toward intertemporal substitution. In his model, the intertemporal elasticity of substitution is not required to be the inverse of the constant coefficient of risk aversion, as is the case for power utility functions. He concludes that, despite the extra degree of freedom, the model cannot explain the economics puzzles, and argues that the existence of heterogeneity and of serious market imperfections is necessary in order to attempt to explain the economy. The author further claims that *"Under the expected time-additive utility restriction $\rho = \gamma$, decreasing the intertemporal elasticity of substitution [inverse of ρ] amounts to increasing the coefficient of relative risk aversion [γ], and results in the simultaneous rise of the risk premium and the risk-free rate."*

I claim that this statement is not correct, as shown by Equation 3 and can be seen from Figures 1 and 2. (The results are practically the same regardless of continuous time or discrete time modelling.) The interest rate is an increasing function of risk aversion for $p \leq \alpha/\sigma^2 - \frac{1}{2}$,

[1] One of the weak points of this argument is the implicit assumption of frictionless market, where people can borrow at risk free rate. However, it is true that the same conclusion holds even for high bid/ask spread for borrowing and lending

[2] The precautionary savings effect is not an artifact of power utility functions. The effect with the same order exists for, for example, exponential utility with constant absolute risk aversion. Intuition suggests that any reasonable utility function should exhibit the precautionary savings behavior (negative third derivative of utility function), as with more volatile consumption people would tend to save more since they are worried about low consumption states. With extra motivation for savings the shadow interest rate must decrease.

while it is a decreasing function of risk aversion for greater p, as can be easily seen by taking the derivative in (3). In Table 1 we can see for different value of expected growth rate α and volatility σ, the threshold size of p, for which the risk free is already low (equals β). The size of p that maximizes the risk free rate is exactly one half of the values in the table.

(Weil 1989) further claims: *"There is no way to fit both the level of the risk-free rate and the risk premium when the VNM [Von Neumann-Morgenstern utility] restrictions is imposed."* I show next that this conclusion is not necessarily correct either, since it is enough to consider high but realistic risk aversion of agents.[3]

From Figure 1 we can see that no "low risk free rate puzzle" is present for expected consumption growth of 3% and volatility of consumption of 2%, with high risk aversion coefficient close to $p = 150$. For example, if the representative agent's risk aversion is $p = 149$, we get $r = \beta$. For $p = 150$ we already have $r = \beta - 3\%$. For higher standard deviation of consumption of 2.5%, Table 1 shows that the "borderline" risk aversion is 95, for which $r = \beta$. In (Mehra and Prescott 1985) the authors use the historical average consumption growth of 1.83% with standard deviation of 3.57%. In this case, the "borderline" risk aversion is just $p = 27.7$. However, modern data indicate that the actual volatility of consumption is lower.

2.1 High sensitivity of interest rate to p

An argument could be made that with high coefficients of risk aversion, the size of interest rate, as a quadratic function of p, is very sensitive to small changes in p. Thus, other things being equal, a small change in consumers' attitude towards risk causes a large change in interest rate. For example, with $\alpha = 3\%$, $\sigma = 2\%$, $\beta = 5\%$, the size of shadow risk free rate for $p = 140$ is 30.2%, while it is -30.2% for $p = 160$. Similarly small changes in volatility would, other things being equal, cause enormous changes in the risk free rate, which does not move that much in reality.

[3]In (Weil 1989) the author argues that a realistic coefficient of relative risk aversion must be in the range of 1 to 5, probably close to 1. A simple argument of draw-down probabilities in (Janeček 2002) shows that it is utterly implausible to consider risk aversion close to 1. While the draw-down probability argument may not apply for risk aversions greater than 4, other considerations show that it is implausible to assume coefficients of risk aversion lower than 20 *for the purposes of taking investment risks.*

The problem of this argument is the assumption of *other things being equal*. The point is that small changes in attitude towards risk should, at the first place, cause *small* changes in the volatility of consumption, possibly also changes in the size of consumption growth. Similarly, volatility of consumption may be a result of equilibrium, where higher risk aversion implies lower volatility.

In this context it is interesting to note the slope of the risk free interest rate at the threshold point for which $r = \beta$. By taking derivative of (3) we get

$$r'(p) = \alpha - \left(p + \frac{1}{2}\right)\sigma^2.$$

For the large value of p that solves equation $r(p) = \beta$ we have

$$p_0 = \frac{2\alpha}{\sigma^2} - 1.$$

Thus, the slope at p_0 equals

$$r'(p_0) = -\alpha + \frac{1}{2}\sigma^2 \approx -\alpha,$$

since σ^2 is small. An increase in p by 1 causes a drop in r by α percent, other things being equal.

3 Conclusion

In this paper I showed that the puzzle of low risk free rate can be resolved by assuming the existence of standard utility function, as soon as the corresponding risk aversion coefficient is sufficiently high. The low risk free rate is explained by the precautionary savings effect, which exists due to the volatility of consumption. This effect may have been underestimated by economists, since few people would consider high coefficients of relative risk aversion in conjunction with the fact that the size of precautionary savings is a quadratic function of p.

To understand why the high risk aversion coefficient for explaining the consumption data (and consequently the size of risk free rate) is plausible, it is important to note that the risk aversion for consumption growth and volatility of consumption is influenced by each consumer in the economy. Thus, the risk aversion should be significantly larger than the risk aversion of investors in financial markets, who are a special selected group.

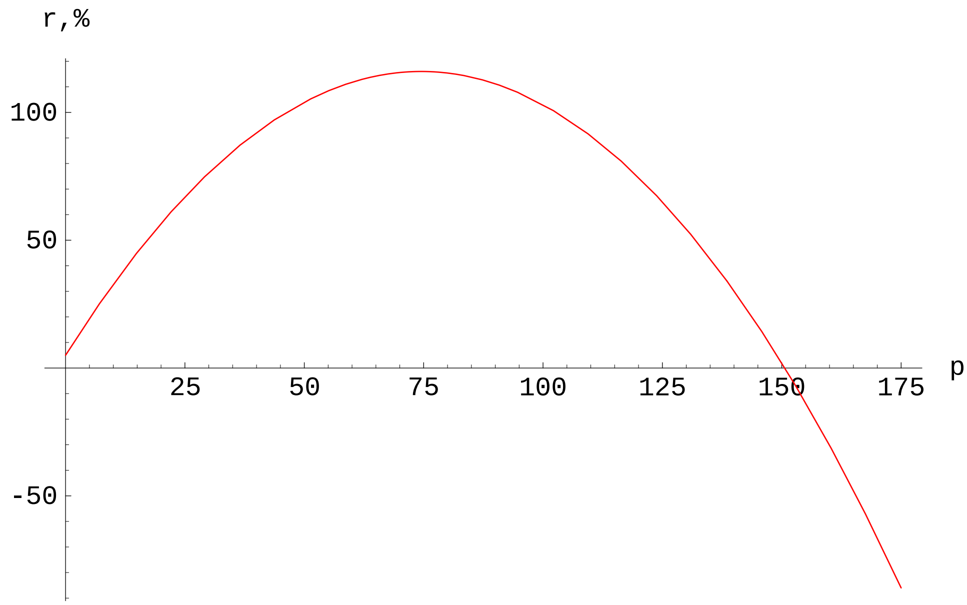

Figure 1: Size of risk free rate, with $\alpha = 3\%$, $\sigma = 2\%$, $\beta = 5\%$.

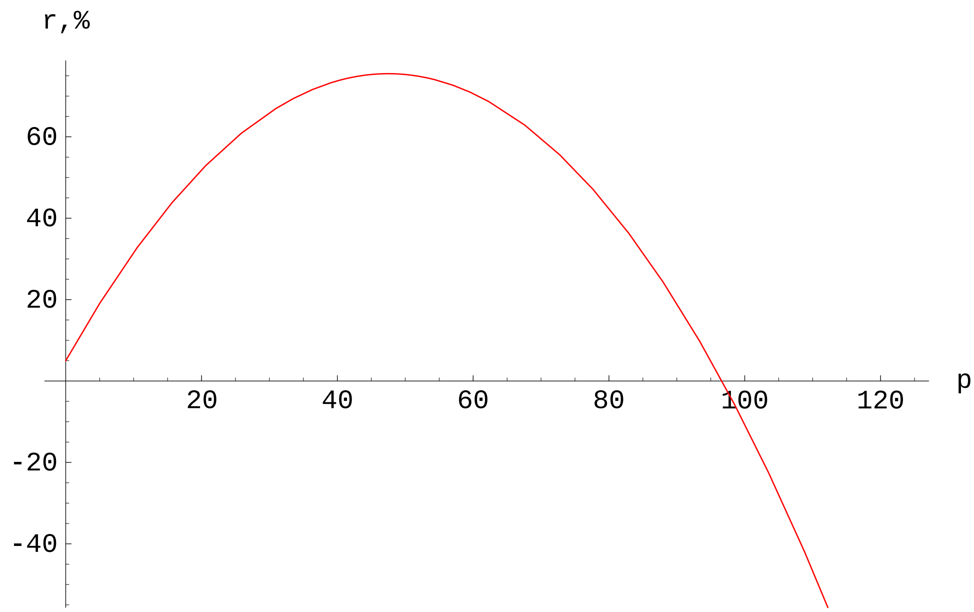

Figure 2: Size of risk free rate, with $\alpha = 3\%$, $\sigma = 2.5\%$, $\beta = 5\%$.

$\alpha\backslash\sigma$	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
1.50	132	97	74	58	47	39	32	27	23	20	18
1.75	155	113	87	68	55	45	38	32	28	24	21
2.00	177	130	99	78	63	52	43	37	32	27	24
2.25	199	146	112	88	71	59	49	42	36	31	27
2.50	221	162	124	98	79	65	55	46	40	35	30
2.75	243	179	137	108	87	72	60	51	44	38	33
3.00	266	195	149	118	95	78	66	56	48	42	37
3.25	288	211	162	127	103	85	71	61	52	45	40
3.50	310	228	174	137	111	92	77	65	56	49	43
3.75	332	244	187	147	119	98	82	70	60	52	46
4.00	355	260	199	157	127	105	88	75	64	56	49

Table 1: Threshold p for which r equals to β, for various % values of α and σ.

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