Optimal investment with high-watermark fee in a multi-dimensional jump-diffusion model

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July 30, 2019

Abstract

We study an optimal investment and consumption problem on infinite horizon, under the assumption that one of the investment opportunities is a fund charging high-watermark fees. The fund and the additional risky assets follow a multi-dimensional geometric Lévy structure. The interest rate is constant and the utility function has constant relative risk aversion (CRRA). Identifying the wealth of the investor together with the *distance to paying fees* as the appropriate states, we obtain a two-dimensional stochastic control problem with both *jumps and reflection.* We derive the Hamilton-Jacobi-Bellman (HJB) integro-differential equation, reduce it to one dimension, and then show it has a smooth solution. Using verifications arguments the optimal strategies are obtainded in feedback form. Some numerical results display the impact of the fees on the investor.

1 Introduction

Hedge-fund managers charge fees for their service. The fee structure usually consists of proportional fees (percentage per year) and *high-watermark* (performance) fees, with the provision that the investor pays a given percentage of the profit (but not losses) made from investing in the fund. The high-watermark is the historic maximum of profits up to date. The performance fees are charged whenever the high-watermark exceeds the previously attained maximum. In the hedge-fund industry, we often see a “2/20 rule”: a 2% per year proportional fee and a 20% high-watermark fee.

We consider here a geometric Lévy multi-dimensional model of risky assets, where one of them is such a fund charging high-watermark fees. Proportional fees can be easily incorporated by subtracting the percentage per year from the rate of return of the fund. The money market pays constant interest. In this market model we study the problem of optimal investment and consumption with a constant relative risk aversion utility function (CRRA) on infinite horizon. With a careful choice of state variables, the mathematical setup becomes a two-dimensional control problem with both jumps and reflection. Using dynamic programming/verification arguments, we find the feedback optimal controls in terms of the smooth solution of the HJB equation. Since the paper may be viewed as a follow up to [18], we describe below the contribution of the present work as well as the relation to the literature.

1.1 Contribution

(i) Generalization of the model: Compared to [18] we allow here for multiple risky assets, *in addition* to the hedge fund. The riskless asset can have non-zero interest rate and we also consider the possible important provision of *hurdles*. The many risky assets (stocks and fund) follow a geometric

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Lévy processes, so they can also have *jumps*. Special care must be taken to assess the high-watermark fees at the time of a jump (in section 2). New mathematical arguments (below) need to be developed to treat this more general model.

(ii) Identification of the state processes: The main modeling challenge is to identify a minimal number of state-processes such that the utility maximization problem can be represented as a Markovian control problem. While the very simple model in [18] could be rather easily framed as a two-dimensional control problem, it is not clear how many state variables we need here. In particular, the attempt to use the wealth process X and the high-watermark N (as in[18]) as a two-dimensional state (X, N) fails. Fortunately (but not obviously) a *still two-dimensional* state process consisting of the cumulative wealth X and the "distance to paying high-watermark fees" Y can be identified. Even more interesting, the state process Y is the solution to the celebrated Skorokhod reflected equation.

(iii) Analytic and numerical solution to a 2-d reflected control problem with jumps: Mathematically, the model (with the identification of states above) leads to a two-dimensional control problem with both jumps and reflection. To the best of our knowledge, a similar control problem is not present in the literature. As one can see below in Section 2, the state can actually jump outside the domain but will be pulled immediately to the boundary. The homogeneity property of the CRRA utility allows us to reduce the dimension of the HJB to one. The reduced HJB equation is an ordinary differential-integral equation, in terms of one variable that is the ratio between the "distance to paying high-watermark fees" and the cumulative wealth. We show that a classical solution of the differential-integral equation exists and can be used to find the optimal solution to our stochastic control problem in *feedback form*. The analysis is a non-trivial generalization of that in [18], based on viscosity solution techniques. Briefly, we first construct a viscosity solution using Perron's method, then we prove smoothness of the solution using properties of viscosity solutions as well as convexity and then we complete the program with a verification argument. There is no closed-form solution to the stochastic control problem so we present some numerical case studies, explaining the quantitative impact of fees (Section 4). The numerical results show that in a scenario of one hedge fund and one stock, the comparison with the case of no fee (the classical Merton problem) is as follows : (i) if the return of the fund is bigger than that of the stock, then the high-watermark fees would make the investor invest *more* in the hedge fund when the high-watermark is close to being reached; (ii) if the return of the fund is smaller than that of the stock, then the high-watermark fees would make the investor invest *less* in the hedge fund when the high-watermark is close to being reached; (iii) in either case, when the investor is far away from paying high-watermark fees, the investment and consumption strategies are close to those in the case of no fee. Note the third comparison result above is also proved analytically. Moreover, the numerics regarding the correlation between the hedge fund and the stock would demonstrate the benefit of diversification, as expected; the effect of jumps in risky assets can be seen as increased volatilities. The details of numerics will be presented in section 4.

1.2 Relation to literature

Utility maximization in continuous time is a fundamental problem in mathematical finance. The seminal work of Merton [23], [24] studied portfolio optimization in a market with one geometric Brownian motion asset, constant interest rate and no frictions. These papers introduced the stochastic control techinques to finance. Various extensions of the models in [23], [24] have been studied in the academic literature. A large body of work (not mentioned here by name) has been devoted to more general market models and utility functions, still under the simplifying assumption of no frictions. Another (closer related to our work) direction of the field considered extensions of the Merton problem with market frictions and imperfections. Some notable examples are transaction costs [9], [32], [10], [30], [25]; or the possibility of bankruptcy [19], [31].

Taking the point of view of the *investor* (as opposed to the fund manager) utility maximization with high-watermark fees is another instance of a Merton problem with market frictions. Mathematically, it produces a more challenging stochastic control problem. Janeček and Sîrbu in [18] proposed an infinite horizon optimal investment and consumption problem, where the only risky asset is a hedge fund charging high-watermark fees at a rate λ, the riskless asset is a bank account charging zero interest, and the utility function is chosen to be power utility. In this model the state process is a *continuous* two-dimensional reflected diffusion. The value function is shown to be a classical solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation and the optimal strategies admit a feedback representation. The problem cannot be solved in closed-form, so [18] provides some numerical results to quantify the effect of high-watermark fees. In the context of the same model, Kontaxis studied in his dissertation [22] asymptotic results for small λ. Lin, Wang and Yao in [34] generalize the model in [18] by assuming the investor is an insurer who is also subject to insurance claims, arriving as a compound Poisson process. The models in [18],[22],[34] assume that the investor can trade continuously in and out of the fund, as we do here as well.

The existing research on high-watermark fees is not limited to the context of an investor optimally rebalancing in and out of a hedge-fund subject to fees, as we do here. Actually, *most of the finance literature* on the topic takes the point of view of the fund manager, assuming the investor leaves a large amount of money with the manager who invests it in in the market. The objective of the manager is to maximize the fees. In [1] and [4], the authors argue that the high-watermark fees serve as incentives for the fund manager to seek long-term growth that is in line with the investor's objective. Panageas and Westerfield [27] studied the problem of maximizing present value of future fees from the perspective of a risk-neutral fund manager. Goetzmann, Ingersoll and Ross [14] derived a closed-form formula for the value of a high-watermark contract as a claim on the investor's wealth. Closest to our work, Guasoni and Obłój [16] perform a mathematical study of a utility maximization problem from the perspective of a hedge fund manager. In [16], the stochastic differential equation governing the evolution of the hedge fund share price has a similar pathwise solution to the state equation describing the dynamics of the investor's wealth in this paper. However, the stochastic control problem is different. Guasoni and Wang [17] study a similar problem to [16] where the fund manager has additional private wealth that can be invested in a different asset. Most recently, [6] provides a closed form solution for the investment strategy of the fund-manager compensated by performance fees.

The problem in [18] and the extension we propose here are technically related to the problem of optimal investment with draw-down constraints in [15], [8], [29] and [11]. However, with consumption present in the running maximum, the problem in [18] does not have a closed-form solution, as opposed to that in [29] and [11]. Obviously, a closed form solution cannot be found in the more general model we consider here. Our analysis relies heavily on the theory of viscosity solutions and Perron's method. We refer the readers to [7], [12], [20] for a quick introduction. Viscosity solutions applied to integro-differential equations are discussed in [2], [3], [28], [5]. For stochastic control problems with jumps, our references include [33], [26].

2 Model

Consider a hedge fund with share price process F and n additional stocks with price processes denoted by $(S^1, \ldots, S^n)$. Together, the $n+1$ risky assets evolve as a multi-dimensional geometric Lévy process:

$$\begin{bmatrix} \frac{dF_t}{F_{t-}} \\ \frac{dS^1_t}{S^1_{t-}} \\ \vdots \\ \frac{dS^n_t}{S^n_{t-}} \end{bmatrix} = \begin{bmatrix} \mu^F \\ \mu^1 \\ \vdots \\ \mu^n \end{bmatrix} dt + \sigma d\mathbf{W}_t + \int_{\mathbb{R}^l} \mathbf{J}\left(\eta\right) \mathcal{N}\left(d\eta, dt\right), \tag{1}$$

where $\sigma\sigma^T > 0$, $\mathbf{W}$ is a d-dimensional Brownian motion, $\mathcal{N}\left(d\eta, dt\right)$ is a Poisson random measure on $\mathbb{R}^l \backslash \{0\} \times [0, \infty)$, with intensity $\mathbf{q}(d\eta)dt$, where $\mathbf{q}$ is σ-finite. All the vectors and matrices are of appropriate dimensions. Both the d-dimensional Brownian motion $\mathbf{W}$ and the measure $\mathcal{N}$ are defined and adapted on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. By the measure $\mathcal{N}$ being adapted we understand that the counting process $\int_0^t \int_A \mathcal{N}(d\eta, ds)$ is adapted for any measurable A with $q(A) < \infty$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. Furthermore, the Brownian motion W and the random measure $\mathcal{N}$ are independent. To ensure price processes to stay positive, we assume that the fund and stock prices never have downside jumps of 100% or more, i.e. $\mathbf{J}$ and $\mathbf{q}$ must satisfy

$$\mathbf{q}\left(\eta | \mathbf{J}(\eta) \in \left((-1,\infty)^{n+1}\right)^c\right) = 0. \tag{2}$$

Moreover, we assume that

$$\int_{\mathbb{R}^l} |\mathbf{J}(\eta)| \mathbf{q}(d\eta) < \infty. \tag{3}$$

and

$$\int_{\mathbb{R}^l} |\mathbf{J}(\eta)|^2 \mathbf{q}(d\eta) < \infty. \tag{4}$$

where $|\cdot|$ denotes some norm on $\mathbb{R}^{n+1}$. Assumption (3) means that the jumps are Lévy processes of finite variation paths. This is not really necessary as long as we compensate all the jumps. We make this assumption in order to simplify our discussion about Ito's formula and HJB equation, which is already cumbersome. To summarize, the right-hand side of (1) is a Lévy process with (square) integrable jumps and adapted to the given filtration.

An investor chooses to invest θ_t^F units of wealth in the hedge fund at time t (right before the jump), where θ^F is a predictable process. The *realized* profit, denoted by P, is subject to both high-watermark and hurdle provisions. In order to impose the hurdle provision, consider a benchmark asset:

$$\frac{dB_t}{B_{t-}} = \mu^B dt + \sigma^B d\mathbf{W}_t + \int_{\mathbb{R}^l} J^B(\eta) \mathcal{N}(d\eta, dt).$$

Investing according to θ^F in the benchmark asset yields the profit $P_t^B = \int_0^t \theta_s^F \frac{dB_s}{B_{s-}}$. The investor is given an initial high-watermark $y \geq 0$ for her profits (in practical applications, we have $y = 0$). The realized profit is reduced by a ratio $\lambda > 0$ of the excess (realized) profit over the strategy of identically investing in the benchmark. In a more mathematical notation, the fees paid to the hedge fund manager in the infinitesimal interval dt amount to λdM_t, where the process M is the so called *high-watermark*

$$M_t \triangleq \sup_{0\leq s\leq t} \left\{ \left(P_s - P_s^B\right) \vee y \right\}.$$

Again, M is the running maximum of the excess *realized* profit from the investment *over* the possible profit from an alternative identical investment in the benchmark With these notations, the *realized* accumulated profit P of the investor evolves as

$$\begin{cases} dP_t = \theta_t^F \frac{dF_t}{F_{t-}} - \lambda dM_t, \ P_{0-} = 0, \\ M_t = \sup_{0\leq s\leq t} \left\{ \left(P_s - P_s^B\right) \vee y \right\}. \end{cases} \tag{5}$$

Equation (5) is implicit, so the existence and uniqueness of the solution should be analyzed carefully. Fortunately, we can solve (P, M) *closed form pathwise*, as shown in Proposition 2.1 below. The investor can also invest in the n stocks whose share prices are given by S_i, $i = 1, \ldots, n$. The investor chooses to invest θ_t^i units of her wealth in stock i at time t, and also to consume at a rate γ_t per unit of time. The remaining wealth sits in a bank account paying interest rate r. With (P, M) denoting the solution to (5), the total wealth of the investor evolves as

$$\begin{cases} dX_t = r\left(X_t - \theta_t^F - \sum_{i=1}^n \theta_t^i\right) dt - \gamma_t dt + \sum_{i=1}^n \theta_t^i \frac{dS_t^i}{S_{t-}^i} + \underbrace{\theta_t^F \frac{dF_t}{F_{t-}} - \lambda dM_t}_{dP_t}, \\ \\ X_{0-} = x. \end{cases} \tag{6}$$

Again from Proposition 2.1 below, the state equation (6) can be solved *pathwise and closed-form* in terms of $(\theta^i, \gamma), i = F, 1, \ldots, n$ provided all stochastic integrals involved are well defined. In addition, for $x > 0$ and $y \geq 0$ we impose the constraints on the set of controls $(\theta^i, \gamma), i = F, 1, \ldots, n$ such that neither shorting selling of hedge fund shares or stocks, nor borrowing from money market is allowed (see Remark 2.1 below), and we will address admissibility in detail below. The investor has homogeneous preferences, meaning that the utility function U has the particular form

$$U(\gamma) = \frac{\gamma^{1-p}}{1-p}, \ \gamma > 0,$$

for some $p > 0, p \neq 1$ called the relative risk-aversion coefficient. For a discount factor $\beta > 0$, the goal is to maximize the expected utility from consumption $E\left[\int_0^\infty e^{-\beta t} U(\gamma_t)\, dt\right]$ over all choices of (θ, c) that keep wealth positive.

$$(\Theta, c) \text{ such that } X > 0 \to \operatorname{argmax} E\left[\int_0^\infty e^{-\beta t} U(\gamma_t)\, dt\right]. \tag{7}$$

Remark 2.1. Since $X_t > 0$ (so $X_{t-} > 0$ as well), admissible strategies can be equivalently represented in terms of the proportions $\pi = \theta / X_-$ and $c = \gamma / X_-$. In this representation we impose the constraint that $\pi_t^i \geq 0, i = F, 1, \ldots, n$ and $\pi_t^F + \sum_1^n \pi_t^i \leq 1$ for all times t. This means there is neither short selling of hedge fund shares or stocks, nor borrowing from money market. In other words, $\pi \in \Delta = \{\pi^i \geq 0, i = F, 1, \ldots, n \text{ and } \pi^F + \sum_{i=1}^n \pi^i \leq 1\}$. While the constraints have clear financial meaning, we actually impose them in order to be able to treat the case of jumps in a tractable manner. This constraint makes admissibility universal, for any $\mathbf{q}$ and $\mathbf{J}$ (see the short comment after equation (13)).

We now turn our attention to the solution of the implicit equation (5).

Remark 2.2. [The Skorokhod reflection mapping] Consider $i \geq 0$ and a right-continuous with left-limits function (path) $f : [0, \infty) \to \mathbb{R}$ with $f(0-) = 0$. Given f, we pose the the implicit reflected equation for the "reflected path" g (right-continuous with left-limits) and the "cumulative reflection" k given formally by the conditions:

1. $g(t) = i + f(t) + k(t) \geq 0$ for all t;
2. k is non-decreasing and right continuous with left limits, $k(0-) = 0$ and,
3. $\int_0^t \mathbf{1}_{\{g(s) > 0\}} dk(s) = 0$ for all t.

The celebrated Skorokhod reflection mapping (see [21] for discontinuous paths) says that the implicit equation above has a unique solution (g, k) given explicitly by the "cumulative reflection"

$$k(t) = \sup_{0 \leq s \leq t} [-f(s) - i]^+ .$$

The "reflected path" g is given explicitly, accordingly.

For our high-watermark model, we use hats for so called "paper quantities", which means quantities computed if no fees are imposed, all else being equal.

Proposition 2.1 (Skorohod reflection and pathwise solutions). *Assume that the predictable process θ_t^F is such that paper profit from investing in F as well as the paper excess profits process corresponding to the trading strategy θ_t^F, namely*

$$\widehat{P}_t := \int_0^t \theta_t^F \frac{dF_t}{F_{t-}}, \quad \widehat{I}_t := \int_0^t \theta_t^F \left(\frac{dF_t}{F_{t-}} - \frac{dB_t}{B_{t-}}\right), \ 0 \leq t < \infty,$$

are well defined stochastic integrals. Recall $P_t^B := \int_0^t \theta_s^F \frac{dB_s}{B_{s-}}$. Then

1. *a couple* (P, M) *is a solution to* (5) *if and only if the "distance to paying high-watermark fees" process*

$$Y_t := M_t - \left(P_t - P_t^B\right)$$

together with the running maximum M *satisfy the Skorohod reflected equation*

$$\begin{cases} dY_t = -d\widehat{I}_t + (1+\lambda)\, dM_t, \\ Y_{0-} = y, \end{cases} \quad \text{subject to } Y_t \geq 0 \text{ and } \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0, \text{ for all } t.$$

Therefore, $(1+\lambda)(M_t - y)$ *is the solution* k *to the Skorokhod equation above, with* $f(t) = -I_t$ *and* $i = y$.

2. *consequently, the implicit equation* (5) *has a unique pathwise solution defined by can be represented pathwise by*

$$P_t = \widehat{P}_t - \frac{\lambda}{1+\lambda}\left(\widehat{M}_t - y\right), \ M_t = y + \frac{1}{1+\lambda}\left(\widehat{M}_t - y\right), \ 0 \leq t < \infty, \tag{8}$$

with $\widehat{M}$ *being the "paper running max"* $\widehat{M}_t := y + \sup_{0 \leq s \leq t}\left[\widehat{I}_s - y\right]^+, \ 0 \leq t < \infty,$

Proof. The proof of the first part is rather obvious. In order to obtain the second part, we identify

$$i = y, \ f(t) = -\widehat{I}_t, \ (1+\lambda)(M_t - y) = k(t),$$

in Remark 2.2. The connection between the continuous Skorokhod equation and high-watermarks has been mentioned in [22]. □

We want to solve our utility maximization problem using dynamic programming arguments. To keep the analysis tractable we wish to find a state process with minimal dimension. It turns out that "the distance to paying high-watermark fees" process Y above is tailor made for such purpose. Not only that Y is an explicit solution of the Skorohod reflection equation, but the pair (X, Y) (X defined in (6)) is a Markovian 2-dimensional controlled reflected equation with jumps. The domain is $D := \{x > 0, y \geq 0\} \ni (X, Y)$, and the evolution of the 2-dimensional system is:

$$\begin{cases} dX_t = rX_t dt + \theta_t^F \left(\frac{dF_t}{F_{t-}} - rdt\right) + \sum_{i=1}^n \theta_t^i \left(\frac{dS_t^i}{S_{t-}^i} - rdt\right) - \gamma_t dt - \lambda dM_t, \\ dY_t = -\theta_t^F \left(\frac{dF_t}{F_{t-}} - \frac{dB_t}{B_{t-}}\right) + (1+\lambda)\, dM_t, \ \int_0^t \mathbf{1}_{\{Y_s > 0\}} dM_s = 0, \end{cases} \tag{9}$$

with initial conditions $(X_{0-}, Y_{0-}) = (x, y) \in D$. The "reflection" process M is non-decreasing, right-continuous, and "acts" only when $Y = 0$. We will use both a row and column notation for the 2-dimensional state.

We denote by $\mu^E = \mu^F - \mu^B$, $\sigma^E = \sigma^F - \sigma^B$, $J^E(\eta) = J^F(\eta) - J^B(\eta)$ and we also denote by $\theta_t \triangleq \left(\theta_t^F, \theta_t^1, \ldots \theta_t^n\right)^T \in \mathbb{R}^{n+1}$ the complete investment strategy at time t. We define

$$\alpha \triangleq (\alpha_F, \alpha_1, \ldots \alpha_n)^T = (\mu_F - r, \mu_1 - r, \ldots \mu_n - r)^T \in \mathbb{R}^{n+1}.$$

Remark 2.3. Note that α cannot be interpreted as the vector of excess returns because the vector of excess returns is indeed $\alpha + \int_{\mathbb{R}^l} \mathbf{J}(\eta)\, \mathbf{q}(d\eta)$.

We now solve for the process (X, Y). The path-wise representation in Proposition 2.1 can be easily translated into a path-wise solution for (X, Y), i.e. we have the following proposition, whose proof is a direct consequence of Proposition 2.1, so we omit it.

Proposition 2.2. *Assume that the predictable processes (θ,γ) satisfy the following integrability property:*

$$\mathbb{P}\left(\int_0^t \left(|\theta_u|_2^2+\gamma_u\right) du<\infty \quad \forall\, 0\leq t<\infty\right)=1,$$
$$\mathbb{P}\left(\int_0^t \left(\int_{\mathbb{R}^l} \left|\theta_u^F \mathbf{J}^E\left(\eta\right)\right|^2 \mathbf{q}\left(d\eta\right)\right) du<\infty \quad \forall\, 0\leq t<\infty\right)=1,$$
$$\mathbb{P}\left(\int_0^t \left(\int_{\mathbb{R}^l} \left|\theta_u^T \mathbf{J}\left(\eta\right)\right|^2 \mathbf{q}\left(d\eta\right)\right) du<\infty \quad \forall\, 0\leq t<\infty\right)=1.$$

Denote by $\widehat{I}$ the paper profit from investing in the fund, N the (paper) profit from investing in all assets, and C the accumulated consumption:

$$\widehat{I}_t=\int_0^t \theta_u^F\left(\mu^E du+\sigma^E d\mathbf{W}_u+\int_{\mathbb{R}^l}\mathbf{J}^E\left(\eta\right)\mathcal{N}\left(d\eta,du\right)\right),$$
$$N_t=\int_0^t \theta_u^T\left(\alpha du+\sigma d\mathbf{W}_u+\int_{\mathbb{R}^l}\mathbf{J}\left(\eta\right)\mathcal{N}\left(d\eta,du\right)\right),$$
$$C_t=\int_0^t \gamma_u du,\ 0\leq t<\infty.$$

The state system (9) has a unique solution (X,Y), which can be represented by

$$X_t=x+\int_0^t rX_s ds+N_t-C_t-\lambda\left(M_t-y\right),\ 0\leq t<\infty,$$
$$Y_t=y-\widehat{I}_t+\left(1+\lambda\right)\left(M_t-y\right),\ 0\leq t<\infty. \tag{10}$$

where the high-watermark is computed as $M_t=y+\frac{1}{1+\lambda}\sup_{0\leq s\leq t}\left[\widehat{I}_s-y\right]^+$.

The state process $(X,Y)\in D$ is a controlled two-dimensional reflected jump-diffusion. The investor uses the control (θ,γ). The reflection occurs on the line $\{y=0\}$ in the direction

$$\kappa:=\begin{pmatrix}-\lambda\\ 1+\lambda\end{pmatrix}.$$

The "genuine reflection" comes at the rate dM^c, where M is the high-watermark process and M^c its continuous part. The "reflection of jumps" happens when the jump size of the accumulated paper profit is large enough to cause (X,Y) be out of D prior to assessing the fee. At the time of such a large jump, high-watermark fees will be immediately deducted so that the (after-fees) process (X,Y) is pulled back to the line $\{y=0\}$ in the direction κ, as in Figure 1.

Remark 2.4. We observe that

$$\int_0^t \mathbf{1}_{\{Y_{s-}\neq 0\}\cup\{Y_s\neq 0\}} dM_s^c=0,$$

which means dM_t^c is a measure only supported on $\{Y_{s-}=Y_s=0\}$. This is true because even if there are diffusion reflections immediately after jump reflections, there are a countable number of jumps.

Remark 2.5. Figure 1 only show the jumps coming from the hedge fund. There may be other simultaneous jumps coming from the other stocks. We can think of simultaneous jumps as a sequence: jump from fund, and immediately jumps from stocks. It is straightforward that jumps from stocks would cause a shift of X while having no effect on Y.

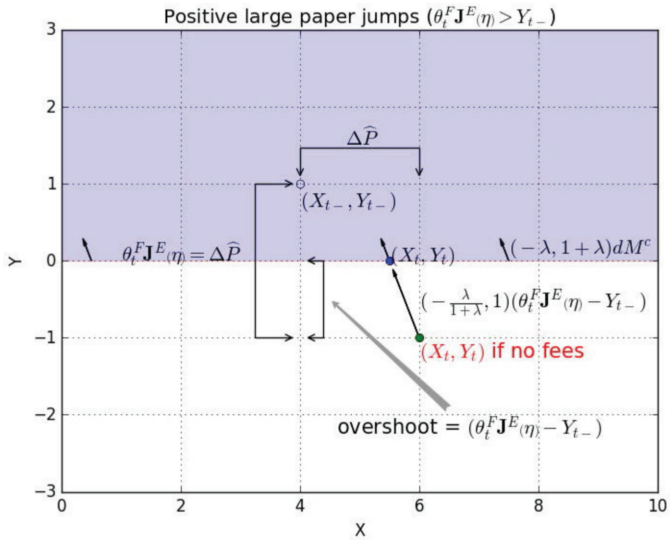

Figure 1: Positive large paper jumps ($\theta_t^F \mathbf{J}^E(\eta) > Y_{t-}$)

Admissibility: Define

$$\mathcal{A}(x,y) := \left\{ \begin{array}{c} (\theta, \gamma) : \\ \text{Predictable processes satisfying integrability in Prop 2.2;} \\ \pi_t = \frac{\theta_t}{X_{t-}} \in \Delta; \\ \gamma_t \geq 0, X_t > 0 \text{ for all } t \geq 0. \end{array} \right\} \tag{11}$$

Value function:

$$V(x,y) \triangleq \sup_{(\theta,\gamma)\in\mathcal{A}(x,y)} E\left[\int_0^\infty e^{-\beta t} U(\gamma_t)\, dt\right],\ x > 0, y \geq 0. \tag{12}$$

Recall from Remark 2.1 that $\Delta = \{\pi^i \geq 0, i = F, 1, \ldots, n \text{ and } \pi^F + \sum_{i=1}^n \pi^i \leq 1\}$ and the constraint $\pi_t = \frac{\theta_t}{X_{t-}} \in \Delta$ is imposed. In order for X to stay positive all the time, in general we would need π to satisfy

$$\mathbf{q}\left(\pi^T \mathbf{J}(\eta) \leq -1\right) = 0. \tag{13}$$

However, this constraint of π depends on the choice of $\mathbf{q}$ and $\mathbf{J}$. In our model, we impose the universal constraint $\pi \in \Delta$. Because $\pi \in \Delta$ together with our assumption in (2) is sufficient for (13) to hold for any $\mathbf{q}$ and $\mathbf{J}$.

Remark 2.6. In our model, we can easily incorporate the case when, in addition to the proportional high-watermark fee λ, the investor pays a continuous proportional fee with size $\nu > 0$ (percentage of wealth under investment management per unit of time). In order to do this we just need to reduce the size of α by the proportional fee to $\alpha - \nu$ in the evolution of the fund share price.

Finally, for technical reasons, we also assume that $\mathbf{J}$ and $\mathbf{q}$ satisfy

$$\begin{gathered} \max_{\pi\in\Delta} \int_{\mathbb{R}^l} \left| \left(1 + \pi^T \mathbf{J}(\eta)\right)^{1-p} - 1 \right| \mathbf{q}(d\eta) < \infty, \\ \max_{\pi\in\Delta} \int_{\mathbb{R}^l} \left| \left(1 + \pi^T \mathbf{J}(\eta)\right)^{-p} \pi^T \mathbf{J}(\eta) \right| \mathbf{q}(d\eta) < \infty. \end{gathered} \tag{14}$$

This assumption above ensures that the integral term of the HJB equation (which we will see later) is well-defined, and will also be used in the proof of verification later.

3 Dynamic programming and main results

3.1 Formal derivation of the Hamilton-Jacobi-Bellman(HJB) equation

We denote by $\mathbf{e}_F$ the column vector of dimension $n+1$ with a one in the first coordinate and all else zero. We also recall the notation κ for the direction of the two dimensional reflection:

$$\mathbf{e}_F := (1, 0, ...0)^T \in \mathbb{R}^{n+1}, \quad \kappa := (-\lambda, 1+\lambda)^T \in \mathbb{R}^2.$$

We also make the mapping notations: $\mathbf{b} : \mathbb{R}^2 \to \mathbb{R}^{n+1}$ and $\mathbf{A} : \mathbb{R}^{2,2} \to \mathbb{R}^{(n+1),(n+1)}$ defined as

$$\mathbf{b}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := x_1\alpha - x_2\mu^E\mathbf{e}_F$$

$$\mathbf{A}\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} := y_{11}\sigma\sigma^T - y_{12}\sigma\sigma^E\mathbf{e}_F^T - y_{21}\mathbf{e}_F\left(\sigma^E\right)^T\sigma^T + y_{22}\mathbf{e}_F\left(\sigma^E\right)^T\sigma^E\mathbf{e}_F^T.$$

Assume the state process sits at position $(X_{t-}, Y_{t-}) = (x, y)$ while the vector position is $\theta_t = \theta$ in the risky assets. In case a "point" of size η comes into the Poisson measure $\mathcal{N}$ exactly at that time, then, before and after the jump we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X_{t-} \\ Y_{t-} \end{pmatrix} \longrightarrow \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} x + \theta^T\mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\theta^F\mathbf{J}^E(\eta) - y\right]^+ \\ y - \theta^F\mathbf{J}^E(\eta) + \left[\theta^F\mathbf{J}^E(\eta) - y\right]^+ \end{pmatrix}.$$

For any function $v : D \to \mathbb{R}$ we, therefore, denote by $\mathbf{U}v(x, y, \theta, \eta)$ the jump of the process $v(X_t, Y_t)$ at that time, which is

$$\mathbf{U}v(x, y, \theta, \eta) := v\begin{pmatrix} x + \theta^T\mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\theta^F\mathbf{J}^E(\eta) - y\right]^+ \\ y - \theta^F\mathbf{J}^E(\eta) + \left[\theta^F\mathbf{J}^E(\eta) - y\right]^+ \end{pmatrix} - v\begin{pmatrix} x \\ y \end{pmatrix}. \tag{15}$$

Lemma 3.1. *Let (X, Y, M) denote the solution of the state equation (10) for fixed $(\theta, \gamma) \in \mathcal{A}(x, y)$. If v is a C^2 (up to the boundary) function on $\left\{(x, y) \in \mathbb{R}^2 : x > 0, y \geq 0\right\}$, and assuming integrability condition below in (16), then*

$$\begin{aligned}
&\int_0^t e^{-\beta s}U(\gamma_s)\,ds + e^{-\beta t}v(X_t, Y_t) = v(x, y) \\
&+ \int_0^t e^{-\beta s}\left\{\begin{array}{l} -\beta v(X_{s-}, Y_{s-}) + U(\gamma_s) + (rX_s - \gamma_s)\,v_x(X_{s-}, Y_{s-}) \\ +\mathbf{b}^T\left(\mathbf{D}v(X_{s-}, Y_{s-})\right)\theta_s + \frac{1}{2}\theta_s^T\mathbf{A}\left(\mathbf{D}^2v(X_{s-}, Y_{s-})\right)\theta_s \end{array}\right\}ds \\
&+ \int_0^t e^{-\beta s}\left\{\kappa^T\mathbf{D}v(X_{s-}, Y_{s-})\right\}dM_s^c \\
&+ \int_0^t e^{-\beta s}\left\{v_x(X_{s-}, Y_{s-})\,\theta_s^T\sigma - v_y(X_{s-}, Y_{s-})\,\theta_s^F\left(\sigma^E\right)^T\right\}d\mathbf{W}_s \\
&+ \int_0^t\int_{\mathbb{R}^l} e^{-\beta s}\mathbf{U}v(X_{s-}, Y_{s-}, \theta_s, \eta)\mathcal{N}(d\eta, ds).
\end{aligned}$$

The compensated process

$$\int_0^t\int_{\mathbb{R}^l} e^{-\beta s}\mathbf{U}v(X_{s-}, Y_{s-}, \theta_s, \eta)\mathcal{N}(d\eta, ds) - \int_0^t\int_{\mathbb{R}^l} e^{-\beta s}\mathbf{U}v(X_{s-}, Y_{s-}, \theta_s, \eta)\mathbf{q}(d\eta)\,ds$$

is a local martingale, in the case that the following condition holds,

$$\int_0^t\int_{\mathbb{R}^l} e^{-\beta s}\left|\mathbf{U}v(X_{s-}, Y_{s-}, \theta_s, \eta)\right|\mathbf{q}(d\eta)\,ds < \infty, \ \ a.s. \tag{16}$$

Remark 3.1. As mentioned, the assumption (3) is not necessary as long as we compensate the jump terms. If we did so, we would introduce extra derivative terms, i.e., v_x, v_y, in the non-local part in the HJB equation (22). The Itô's lemma above is already notationally cumbersome, so we insist on not compensating the jump term for simplicity. Our analysis based on viscosity solutions would still apply in the more general (compensated) case with an appropriate definition of viscosity solutions.

Recall that M was explicitly defined in (2.2). Taking into account that dM_t^c is a measure with support on the set of times $\{t \geq 0 : Y_{t-} = Y_t = 0\}$, we can *formally* write down the HJB equation:

$$\sup_{\gamma \geq 0, \frac{\theta}{x} \in \Delta} \left\{ -\beta v + U(\gamma) + (rx - \gamma) v_x + \mathbf{b}^T (\mathbf{D}v) \theta + \frac{1}{2} \theta^T \mathbf{A} \left(\mathbf{D}^2 v\right) \theta + \int_{\mathbb{R}^l} \mathbf{U} v(x, y, \theta, \eta) \mathbf{q}(d\eta) \right\} = 0$$

The HJB should hold in the interior of D, i.e. for $x > 0, y > 0$. The boundary condition is

$$\kappa^T \mathbf{D} v = 0, \quad x > 0, y = 0.$$

Formally, if the HJB equation above, has a smooth solution, the optimal consumption will be given in feedback form by

$$\widehat{\gamma}(x, y) = I(v_x(x, y)), \tag{17}$$

where $I \triangleq (U')^{-1}$ is the inverse of marginal utility. In addition, we expect the optimal investment strategy $\widehat{\theta}$ to be given by

$$\widehat{\theta}(x, y) = \arg\max_{\frac{\theta}{x} \in \Delta} \left\{ \ \mathbf{b}^T (\mathbf{D}v) \theta + \tfrac{1}{2} \theta^T \mathbf{A} \left(\mathbf{D}^2 v\right) \theta + \textstyle\int_{\mathbb{R}^l} \mathbf{J} v(x, y, \theta, \eta) \mathbf{q}(d\eta) \ \right\}. \tag{18}$$

Finally, the smooth solution of the HJB equation should be equal the value function i.e. $v = V$ where V was defined in (12).

3.2 Dimension reduction

We use the *homogeneity property* for the power utility function, with the expectation that

$$v(x, y) = x^{1-p} v\left(1, \frac{y}{x}\right) =: x^{1-p} u(z) \ \text{ for } z := \frac{y}{x}.$$

The optimal amounts $\widehat{\theta}(x, y)$ and $\widehat{\gamma}(x, y)$ in (18) and (17) translate into the *proportions*

$$\widehat{c}(x, y) = \frac{\widehat{\gamma}(x, y)}{x} = \frac{I(v_x(x, y))}{x}, \tag{19}$$

and

$$\widehat{\pi}(x, y) = \frac{\widehat{\theta}(x, y)}{x} \in \Delta. \tag{20}$$

Since

$$\begin{aligned}
v_x(x, y) &= \big((1-p) u(z) - z u'(z)\big) \cdot x^{-p}, \\
v_y(x, y) &= u'(z) \cdot x^{-p}, \\
v_{xx}(x, y) &= \big(-p(1-p) u(z) + 2pz u'(z) + z^2 u''(z)\big) \cdot x^{-1-p}, \\
v_{yy}(x, y) &= u''(z) \cdot x^{-1-p}, \\
v_{xy}(x, y) &= \big(-p u'(z) - z u''(z)\big) \cdot x^{-1-p},
\end{aligned}$$

we define the following differential operators on the function $u(z)$

$$\begin{aligned}
D_x[u](z) &= (1-p) u(z) - z u'(z), \\
D_y[u](z) &= u'(z), \\
D_{xx}[u](z) &= -p(1-p) u(z) + 2pz u'(z) + z^2 u''(z), \\
D_{yy}[u](z) &= u''(z), \\
D_{xy}[u](z) &= D_{yx}[u](z) = -p u'(z) - z u''(z).
\end{aligned}$$

We also reduce the dimension for the the jump part by

$$\begin{aligned}\mathbf{U}v(x,y,\theta,\eta) &= v\left(\begin{array}{c} x+\theta^T\mathbf{J}(\eta)-\frac{\lambda}{1+\lambda}\left[\theta^F\mathbf{J}^E(\eta)-y\right]^+ \\ y-\theta^F\mathbf{J}^E(\eta)+\left[\theta^F\mathbf{J}^E(\eta)-y\right]^+\end{array}\right)-v\left(\begin{array}{c} x \\ y\end{array}\right)\\ &= x^{1-p}\mathbf{R}u(z,\pi,\eta).\end{aligned}$$

The reduced operator $\mathbf{R}$ above acts on functions $u(z), z\geq 0$ as

$$\begin{aligned}&\mathbf{R}u(z,\pi,\eta) := \\ &=\left(1+\pi^T\mathbf{J}(\eta)-\frac{\lambda}{1+\lambda}\left[\pi^F\mathbf{J}^E(\eta)-z\right]^+\right)^{1-p}\cdot u\left(\frac{z-\pi^F\mathbf{J}^E(\eta)+\left[\pi^F\mathbf{J}^E(\eta)-z\right]^+}{1+\pi^T\mathbf{J}(\eta)-\frac{\lambda}{1+\lambda}\left[\pi^F\mathbf{J}^E(\eta)-z\right]^+}\right)-u(z),\end{aligned} \tag{21}$$

with the obvious (vector) substitution $\pi=\theta/x\in\Delta$. We have obtained the reduced one-dimensional HJB equation for $u(z)$, $z>0$ with a boundary condition at $z=0$ (and $z=\infty$, see below):

$$\begin{aligned}&-\beta u+r\cdot D_x[u]+\sup_{c\geq 0}\left\{\frac{c^{1-p}}{1-p}-c\cdot D_x[u]\right\}\\ &\qquad +\sup_{\pi\in\Delta}\left\{\mathcal{B}^T[u]\,\pi+\frac{1}{2}\pi^T\mathcal{A}[u]\,\pi+\int_{\mathbb{R}^l}\mathbf{R}u(z,\pi,\eta)\mathbf{q}(d\eta)\right\}=0,\ z>0\\ &-\lambda(1-p)\,u(0)+(1+\lambda)\,u'(0)=0.\end{aligned} \tag{22}$$

Above,

$$\mathcal{B}[u]:=\mathbf{b}\left(\begin{array}{c} D_x[u]\\ D_y[u]\end{array}\right),\ \text{and}\ \mathcal{A}[u]:=\mathbf{A}\left(\begin{array}{cc} D_{xx}[u] & D_{xy}[u]\\ D_{yx}[u] & D_{yy}[u]\end{array}\right),$$

where $\mathbf{b}$ and $\mathbf{A}$ were defined in Lemma 3.1. Each element of the matrix $\mathcal{A}[u]$ is increasing in u''. This observation will be used several times in our analysis. We expect that

$$\lim_{z\to\infty}u(z)=\frac{1}{1-p}c_0^{-p},$$

with c_0 given by (27) below. The optimal investment proportion in (20) could therefore be expressed (provided we can find a smooth solution for the reduced HJB equation (22)) as

$$\widehat{\pi}(z)=\arg\max_{\pi\in\Delta}\left\{\mathcal{B}^T[u]\,\pi+\frac{1}{2}\pi^T\mathcal{A}[u]\,\pi+\int_{\mathbb{R}^l}\mathbf{R}u(z,\pi,\eta)\mathbf{q}(d\eta)\right\}. \tag{23}$$

If $\widehat{\pi}(z)$ lies in the interior of Δ, we can also use the first order condition to get that $\widehat{\pi}(z)$ satisfies

$$\mathcal{B}[u]+\mathcal{A}[u]\,\pi+\int_{\mathbb{R}^l}\nabla_\pi\mathbf{R}(z,\pi,\eta)\mathbf{q}(d\eta)=0. \tag{24}$$

The optimal consumption proportion $\widehat{c}$ in (19) is formally expressed as

$$\widehat{c}(z)=(D_x[u])^{-\frac{1}{p}}=\left((1-p)\,u(z)-zu'(z)\right)^{-\frac{1}{p}}. \tag{25}$$

3.2.1 The case when paying no fee, $\lambda=0$

This is the classical Merton problem with jumps, with the constraints $\pi_{\mathbf{0}}\in\Delta$. The optimal investment and consumption proportions are constants. We can take the solution from [13], or solve our equation (22) and then use (23) and (25) to obtain the same results. More precisely, for $\lambda=0$, the optimal investment proportions π_0 and the optimal consumption proportion c_0 are given by

$$\pi_0\triangleq\left\{\begin{array}{ll}\arg\max_{\pi\in\Delta}\left\{\begin{array}{l}(1-p)\,\alpha^T\pi+\frac{1}{2}\left(-p\,(1-p)\right)\pi^T\sigma\sigma^T\pi\\ +\int_{\mathbb{R}^l}\left\{\left(1+\pi^T\mathbf{J}(\eta)\right)^{1-p}-1\right\}\mathbf{q}(d\eta)\end{array}\right\}, & p<1,\\ \arg\min_{\pi\in\Delta}\left\{\begin{array}{l}(1-p)\,\alpha^T\pi+\frac{1}{2}\left(-p\,(1-p)\right)\pi^T\sigma\sigma^T\pi\\ +\int_{\mathbb{R}^l}\left\{\left(1+\pi^T\mathbf{J}(\eta)\right)^{1-p}-1\right\}\mathbf{q}(d\eta)\end{array}\right\}, & p>1,\end{array}\right. \tag{26}$$

$$c_0 = \frac{1}{p}\left(\begin{array}{c} \beta - r(1-p) - (1-p)\,\alpha^T\pi_0 + \frac{1}{2}p(1-p)\,\pi_0^T\sigma\sigma^T\pi_0 \\ -\int_{\mathbb{R}^l}\left\{\left(1+\pi_0^T\mathbf{J}(\eta)\right)^{1-p} - 1\right\}\mathbf{q}(d\eta) \end{array}\right) \tag{27}$$

Because of the constraint $\pi \in \Delta$, the optimal π_0 may be on the boundary, i.e. we may have $(\pi_0)_i = 0$ for some $i \in \{F, 1, \ldots, n\}$ or $\sum_i (\pi_0)_i = 1$. The assumption in (14) guarantees that the integrals with respect to $\mathbf{q}$ above are well-defined. Finally, the reduced value function is given by the constant

$$u_0 = \frac{1}{1-p}c_0^{-p} \tag{28}$$

The optimal proportions (26) and (27) are compatible with the feedback formulas (23) and (25). An additional constraint needs to be imposed on the parameters in order to obtain a finite value u_0 for $\lambda = 0$. This assumption equivalent to c_0 in (27) being strictly positive. i.e. we assume that

$$\beta > r(1-p) + (1-p)\,\alpha^T\pi_0 - \frac{1}{2}p(1-p)\,\pi_0^T\sigma\sigma^T\pi_0 + \int_{\mathbb{R}^l}\left\{\left(1+\pi_0^T\mathbf{J}(\eta)\right)^{1-p} - 1\right\}\mathbf{q}(d\eta).$$

In order to compare with the case where there is no investment and only consumption, we also make the following assumption:

$$w_* \triangleq \frac{1}{1-p}\left(\frac{\beta}{p} - r\frac{1-p}{p}\right)^{-p} < u_0. \tag{29}$$

This s equivalent to $\pi_0^i > 0$ for at least one $i \in \{F, 1, \ldots, n\}$, because otherwise we would have $w_* = u_0$. The intuition behind this assumption is that we only consider a portfolio of risky assets worth investing, including the hedge fund share.

3.3 Main results

For fixed $c \geq 0$ and $\pi \in \Delta$, we denote by

$$\mathcal{L}_{c,\pi}[u](z) := \\ -\beta u + r\cdot D_x[u] + \left\{\frac{c^{1-p}}{1-p} - c\cdot D_x[u]\right\} + \left\{\mathcal{B}^T[u]\,\pi + \frac{1}{2}\pi^T\mathcal{A}[u]\,\pi + \int_{\mathbb{R}^l}\mathbf{R}(z,\pi,\eta)d\mathbf{q}(d\eta)\right\}.$$

The reduced HJB equation can be rewritten (with the implicit assumption that $D_x[u] > 0$) as

$$\begin{cases} -\beta u + r\cdot D_x[u] + \widetilde{V}(D_x[u]) + \sup_{\pi\in\Delta}\left\{\mathcal{B}^T[u]\,\pi + \frac{1}{2}\pi^T\mathcal{A}[u]\,\pi + \int_{\mathbb{R}^l}\mathbf{R}(z,\pi,\eta)\mathbf{q}(d\eta)\right\} = 0,\ z > 0, \\ -\lambda(1-p)\,u(0) + (1+\lambda)\,u'(0) = 0, \quad \lim_{z\to\infty} u(z) = \frac{1}{1-p}c_0^{-p}, \end{cases} \tag{30}$$

where $\widetilde{V}(y) = \frac{p}{1-p}y^{\frac{p-1}{p}}$, $y > 0$. The w_* defined in (29) is an important quantity. It satisfies

$$-\beta w_* + r(1-p)\,w_* + \widetilde{V}((1-p)\,w_*) = 0, \text{and } -\beta w + r(1-p)\,w + \widetilde{V}((1-p)\,w) < 0,\ w_* < w \leq u_0.$$

Next theorem shows that the reduced HJB equation (30) has a smooth solution with additional properties.

Theorem 3.2. *There exists a strictly increasing function u which is C^2 on $[0,\infty)$, is a solution to (30), satisfies the condition $u(0) > w_*$ and*

$$(1-p)\,u - zu' > 0,\ \forall z \geq 0,\ \textit{together with } u(z) \to u_0,\ zu'(z),\ z^2u''(z) \to 0 \textit{ as } z \to \infty.$$

The proof of the above theorem is deferred to subsections 3.4 and 3.5. In subsection 3.4 we prove the existence of a viscosity solution using Perron's method, and in subsection 3.5 we upgrade its regularity. Next proposition shows that the so-called closed-loop equation has a unique global solution. Its own proof is postponed to subsection 3.6.

Proposition 3.3. *Fix $x>0, y\geq 0$. Consider the feedback proportions $\widehat{\pi}(z)$ and $\widehat{c}(z)$ defined in (23) and (25), where u is the solution in Theorem 3.2. Define the feedback controls*

$$\widehat{\theta}(x,y) \triangleq x\widehat{\pi}(y/x), \quad \widehat{\gamma}(x,y) \triangleq x\widehat{c}(y/x) \text{ for } x>0, y\geq 0.$$

The closed-loop equation

$$\begin{cases} X_t = & x+\int_0^t rX_s ds + \int_0^t \widehat{\theta}^F(X_{s-},Y_{s-})\left(\frac{dF_s}{F_{s-}}-rds\right) \\ & +\int_0^t \sum_{i=1}^n \widehat{\theta}^i(X_{s-},Y_{s-})\left(\frac{dS_s^i}{S_{s-}^i}-rds\right) - \int_0^t \widehat{\gamma}(X_{s-},Y_{s-})\,ds \quad -\lambda M_t, \\ Y_t = & y-\int_0^t \widehat{\theta}^F(X_{s-},Y_{s-})\left(\frac{dF_s}{F_{s-}}-\frac{dB_s}{B_{s-}}\right)+(1+\lambda)M_t, \\ & \int_0^t \mathbf{1}_{\{Y_s>0\}}dM_s = 0, \end{cases}$$

has a unique strong global solution $\left(\widehat{X},\widehat{Y}\right)$ such that $\widehat{X}>0$ and $\widehat{Y}\geq 0$.

Next theorem addresses the optimality of the feedback controls, with a proof in subsection 3.6.

Theorem 3.4. *Consider the solution u in Theorem 3.2. For each $x>0, y\geq 0$, the feedback proportions $(\widehat{\pi},\widehat{c})$ in (23) and (25) are optimal and*

$$u\left(\frac{y}{x}\right)x^{1-p} \triangleq v(x,y) = V(x,y) \triangleq \sup_{(\theta,\gamma)\in\mathcal{A}(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}U(\gamma_t)\,dt\right].$$

Proposition 3.5 presents some additional properties of the feedback controls. The properties are actually used to prove the existence and uniqueness of the solution to the closed-loop equation in Proposition 3.3. We relegate the proof of this proposition below to the Appendix.

Proposition 3.5. *The feedback controls $\widehat{\pi}$ and $\widehat{c}$ satisfy*

$$0<\widehat{c}(z)\to c_0, \quad 0<\widehat{\pi}(z)\to \pi_0, \quad z\to\infty, \tag{31}$$

and

$$z\widehat{c}'(z)\to 0, \quad z\widehat{\pi}'(z)\to 0, \quad z\to\infty. \tag{32}$$

In addition,

$$\widehat{c}(z)>c_0 \text{ for } z\geq 1 \text{ if } p<1 \text{ and } \widehat{c}(1)<c_0 \text{ if } p>1. \tag{33}$$

3.4 Existence of a viscosity solution

The seminar paper [7] provides a good introduction to viscosity solutions for *local* equations. For the *non-local* equation (22) arising from our model, we adopt a definition of viscosity solutions from [5], though our definitions are slightly less general than that given in [5], since our value function is bounded. To start our definition of viscosity solutions, we consider the general equations written under the form

$$F\left(x,u,\Delta u,D^2u,I[x,u]\right)=0 \text{ in a open domain } \Omega, \tag{34}$$

where F is a continuous function satisfying the local and non-local degenerate ellipticity conditions below in (35). The non-local term $I[x,u]$ can be quite general as seen in [5], a typical form of $I[x,u]$ is

$$I[x,u]=\int_{\mathbb{R}^d}\left(u(x+z)-u(x)-\nabla u(x)\cdot z\mathbf{1}_B(z)\right)\mu(z)$$

for some Lévy measure μ and some ball B centered at 0. The *ellipticity assumption* of F means that for any $x\in\Omega, u\in\mathbb{R}, p\in\mathbb{R}^d, M,N\in\mathbb{S}_d, l_1,l_2\in\mathbb{R}$ we have

$$F(x,u,p,M,l_1[x,u])\leq F(x,u,p,N,l_2[x,u]) \text{ if } M\geq N, l_1\geq l_2, \tag{35}$$

where $\mathbb{S}_d$ denotes the space of real $N\times N$ symmetric matrices. Note that, apart from the usual ellipticity assumption for local equation, $F(x,u,p,M,l)$ is nondecreasing in the non-local operator I. Let us now give a definition of viscosity solutions for the equation (34).

Definition 3.1. An upper semi-continuous and bounded function u is a viscosity subsolution of (34) if, for any bounded test function $\phi \in C^2(\Omega)$, if x is a *global* maximum point of $u - \phi$, then

$$F\left(x, u(x), \Delta\phi(x), D^2\phi(x), I[x, \phi]\right) \leq 0.$$

A lower semi-continuous and bounded function u is a viscosity supersolution of (34) if, for any bounded test function $\phi \in C^2(\Omega)$, if x is a *global* minimum point of $u - \phi$, then

$$F\left(x, u(x), \Delta\phi(x), D^2\phi(x), I[x, \phi]\right) \geq 0.$$

A function u is a viscosity solution of (34) if it is both a subsolution and supersolution.

It is worth mentioning that the boundary conditions used throughout our analysis can be interpreted in the classical sense. Now we turn our attention back to the HJB equation (22). We observe that if $u(z) = u_0$ (defined in (28) in Remark 3.2.1), then

$$\begin{aligned}
& -\beta u_0 + r(1-p)u_0 + \widetilde{V}((1-p)u_0) \\
& + \sup_{\pi\in\Delta}\left\{\begin{array}{c} (1-p)u_0\alpha^T\pi + \frac{1}{2}(-p(1-p)u_0)\pi^T\sigma\sigma^T\pi \\ + \int_{\mathbb{R}^l}\left\{\left(\left(1+\pi^T\mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\pi^F\mathbf{J}^E(\eta) - z\right]^+\right)^{1-p} - 1\right)u_0\right\}\mathbf{q}(d\eta) \end{array}\right\} \\
& \leq -\beta u_0 + r(1-p)u_0 + \widetilde{V}((1-p)u_0) \\
& + \sup_{\pi\in\Delta}\left\{\begin{array}{l} (1-p)u_0\alpha^T\pi + \frac{1}{2}(-p(1-p)u_0)\pi^T\sigma\sigma^T\pi \\ + \int_{\mathbb{R}^l}\left\{\left(\left(1+\pi^T\mathbf{J}(\eta)\right)^{1-p} - 1\right)u_0\right\}\mathbf{q}(d\eta) \end{array}\right\} \\
& = 0,
\end{aligned}$$

and moreover u_0 is actually a classical supersolution of the HJB equation (22), which reads

$$\begin{array}{l} -\beta u + r\cdot D_x[u] + \widetilde{V}(D_x[u]) + \sup_{\pi\in\Delta}\left\{\mathcal{B}^T[u]\pi + \frac{1}{2}\pi^T\mathcal{A}[u]\pi + \int_{\mathbb{R}^l}\mathbf{R}u(z,\pi,\eta)\mathbf{q}(d\eta)\right\} \leq 0,\ z>0, \\ -\lambda(1-p)u(0) + (1+\lambda)u'(0) \leq 0, \quad \lim_{z\to\infty}u(z) \geq \frac{1}{1-p}c_0^{-p}. \end{array} \tag{36}$$

For technical reasons we need a subsolution related to the critical value w_* defined in (29).

Proposition 3.6. *There exists a value $z_* \in (0,\infty)$ and a function*

$$u_s \in C^1[0,\infty) \cap C^2(0, z_*] \cap C^2[z_*, \infty)$$

such that $w_* - \xi \leq u_s \leq u_0$ for some $\xi > 0$ (arbitrarily small) and satisfying

$$\sup_{c\geq 0, \pi\in\Delta} \mathcal{L}_{c,\pi}u_s > 0$$

in the viscosity sense on $(0,\infty)$ (i.e u_s is a strict viscosity subsolution for (30)), and

$$-\lambda(1-p)u_s(0) + (1+\lambda)u_s'(0) > 0, \quad \lim_{z\to\infty}u(z) < \frac{1}{1-p}c_0^{-p}.$$

Proof. For a such that $\frac{\lambda}{1+\lambda}(1-p)w_* < a < (1-p)w_*$, we consider the function

$$u_s(z) = \begin{cases} w_* - \xi + az - \frac{2a}{1+\varepsilon}z^{1+\varepsilon}, & 0 \leq z \leq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}} \\ w_* - \xi + a\frac{e}{1+\varepsilon}\left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}, & z \geq \left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}} \end{cases}$$

for some small $\xi > 0$. We use

$$\begin{aligned}
& \widetilde{V}\left((1-p)\left(w_* - \xi + az - \frac{2a}{1+\varepsilon}z^{1+\varepsilon}\right) - a\left(z - 2z^{1+\varepsilon}\right)\right) \\
& \geq \widetilde{V}((1-p)(w_* - \xi)) + \widetilde{V'}((1-p)(w_* - \xi)) \\
& \cdot\left((1-p)\left(az - \frac{2a}{1+\varepsilon}z^{1+\varepsilon}\right) - a\left(z - 2z^{1+\varepsilon}\right)\right),
\end{aligned}$$

to obtain

$$
\begin{aligned}
&\sup_{c\geq 0,\pi\in\Delta} \mathcal{L}_{c,\pi}u_s(z) \\
&\geq -\beta(w_*-\xi)+r(1-p)(w_*-\xi)+\widetilde{V}((1-p)(w_*-\xi)) \\
&-\beta\left(az-\frac{2a}{1+\varepsilon}z^{1+\varepsilon}\right)+r\left((1-p)\left(az-\frac{2a}{1+\varepsilon}z^{1+\varepsilon}\right)-a\left(z-2z^{1+\varepsilon}\right)\right) \\
&+\widetilde{V'}((1-p)(w_*-\xi))\left((1-p)\left(az-\frac{2a}{1+\varepsilon}z^{1+\varepsilon}\right)-a\left(z-2z^{1+\varepsilon}\right)\right) \\
&+\sup_{\pi\in\Delta}\left\{\mathcal{B}^T[u_s]\pi+\frac{1}{2}\pi^T\mathcal{A}[u_s]\pi+\int_{\mathbb{R}^l}\mathbf{R}u_s(z,\pi,\eta)\mathbf{q}(d\eta)\right\} \\
&\geq -\beta(w_*-\xi)+r(1-p)(w_*-\xi)+\widetilde{V}((1-p)(w_*-\xi)) \\
&-Cz-Dz^{1+\varepsilon}
\end{aligned}
$$

where the last inequality follows from setting $\pi=\mathbf{0}$, and C,D are some constants. For $\xi>0$ fixed,

$$-\beta(w_*-\xi)+r(1-p)(w_*-\xi)+\widetilde{V}((1-p)(w_*-\xi))>0.$$

So, if ε is sufficiently small,

$$
\begin{aligned}
&-\beta(w_*-\xi)+r(1-p)(w_*-\xi)+\widetilde{V}((1-p)(w_*-\xi)) \\
&-C(z-1)-D(z-1)^{1+\varepsilon} \\
&\geq -\beta(w_*-\xi)+r(1-p)(w_*-\xi)+\widetilde{V}((1-p)(w_*-\xi)) \\
&-|C|(z-1)-|D|(z-1)^{1+\varepsilon} \\
&>0 \text{ for } \forall\ 0<z\leq\left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}.
\end{aligned}
$$

Therefore, for such an ε we will have

$$\sup_{c\geq 0,\pi\in\Delta}\mathcal{L}_{c,\pi}u_s(z)>0,\ 0<z\leq\left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}.$$

Since u_s is constant for $z\geq\left(\frac{1}{2}\right)^{\frac{1}{\varepsilon}}$ and is extended to be C^1 we obtain

$$\sup_{c\geq 0,\pi\in\Delta}\mathcal{L}_{c,\pi}u_s(z)>0,\quad z>0,$$

in the viscosity sense and actually in the classical sense for any $z\neq z_*:=(1/2)^{\frac{1}{\varepsilon}}$. □

We now construct a viscosity solution of the HJB equation (30) using Perron's method. We denote by $\mathcal{S}$ the set of functions $h:[0,\infty)\to\mathbb{R}$, which satisfy the following properties:

1. h is continuous on $[0,\infty)$.
2. The function $(x,y)\to x^{1-p}h(y/x)$ is both concave and nondecreasing in the direction $\kappa=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ (from upper left to lower right) within its domain $x>0,y\geq 0$; for fixed x, the function $y\to x^{1-p}h(y/x)$ is concave and nondecreasing in $y\geq 0$.
3. h is a viscosity supersolution of the HJB equation on the open interval $(0,\infty)$.
4. $-\lambda(1-p)h(0)+(1+\lambda)h'(0)\leq 0$.

5. $u_s \le h \le u_0$.

Remark 3.2. Note that 2 and 5 above would imply that $h(z), h(z-), h(z+)$ are bounded. Together with the technical assumption in (14), it ensures that when plugging h into the HJB equation, the integral term is well-defined.

Theorem 3.7. *Define $u := \inf\{h,\ h \in \mathcal{S}\}$. Then, $u_s \le h \le u_0$ is continuous on $[0,\infty)$, is a viscosity solution of the HJB equation on the open interval $(0,\infty)$, and satisfies $-\lambda(1-p)h(0)+(1+\lambda)h'(0)=0$. In addition, The function $(x,y)\to x^{1-p}u(y/x)$ is concave and nondecreasing in the direction $\kappa=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ (from upper left to lower right) within its domain $x>0, y\ge 0$. For fixed x, the function $y\to x^{1-p}u(y/x)$ is concave and nondecreasing in $y\ge 0$, and $u(1)>w_*$.*

Remark 3.3. As a consequence of our construction in Theorem 3.7, we have $u(z)\le u_0(z)$ and therefore $v(x,y)\le v_0(x)$. This means that, with high-watermark fees, the value function is always smaller than the value function of the Merton problem without fees ($\lambda=0$). The intuition for this is rather obvious.

Proof. We follow the ideas of the proof of Proposition 1 in [2], with necessary modifications to take into account the boundary condition at $z=0$ and to keep track of the convexity properties.

1. By construction, as an infimum of concave nondecreasing functions, we have that the function $(x,y)\to x^{1-p}u(y/x)$ is concave and nondecreasing in the direction $\kappa=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ (from upper left to lower right) within its domain $x>0, y\ge 0$, and for fixed x, the function $y\to x^{1-p}u(y/x)$ is concave and nondecreasing in $y\ge 0$.

2. Since $x\to x^{1-p}u(y/x)$ is concave and nondecreasing in the direction $\kappa=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ we conclude that $x\to x^{1-p}u(y/x)$ is continuous in the direction $\kappa=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$, which translates to that u is continuous in $[0,\infty)$.

3. We suppose that a C^2 function φ touches u from below at an interior point $z\in(0,\infty)$. For fixed c,π, each $h\in\mathcal{S}$ is a viscosity supersolution of $\mathcal{L}_{c,\pi}h\le 0$, so by taking the infimum over $h\in\mathcal{S}$ we still get a supersolution, according to Proposition 1 in [2]. In other words, $(\mathcal{L}_{c,\pi}\varphi)(z)\le 0$, and then we can take the supremum over (c,π) to get that u is a supersolution of the HJB equation.

4. By construction, $u_s\le u\le u_0$.

5. For each $h\in\mathcal{S}$, we have $h'(0)\le\frac{\lambda}{1+\lambda}(1-p)h(0)$ which translates in terms of $g(x,y):=x^{1-p}h(y/x)$ as

$$\nabla_\kappa g(1,0)=\frac{\sqrt{2}}{2}\left((1-p)h(0)-h'(0)\right)\ge\frac{\sqrt{2}}{2}\frac{1}{1+\lambda}(1-p)h(0).$$

Taking into account the concavity of $g(x,y)$ along the line $x-y=1$ within its domain $x>0, y\ge 0$, this is equivalent to

$$g(1-\xi,\xi)=(1-\xi)^{1-p}h(\xi/(1-\xi))\le\frac{\sqrt{2}}{2}\frac{1}{1+\lambda}(1-p)h(0)\cdot\sqrt{2}\xi+h(0),\quad 0\le\xi<1.$$

Since (**??**) holds for each $h\in\mathcal{S}$ the same inequality will hold for the infimum, which means that u satisfies (**??**), which reads

$$u'(0)\le\frac{\lambda}{1+\lambda}(1-p)u(0).$$

Let us show that u is a viscosity subsolution. We start by making the following simple observation on the function u: By construction, the function $(x,y)\to x^{1-p}u(y/x)$ is concave in the direction

$\kappa = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ within its domain $x > 0, y \geq 0$. We denote the one-sided directional derivative by $\triangledown_{\kappa-} v$ ($\triangledown_{\kappa+} v$) when (x,y) approaching from upper left to lower right (lower right to upper left). Since

$$\triangledown_{\kappa-} v\left(x,y\right) = \frac{\sqrt{2}}{2} x^{-p} \cdot \left(\left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z+\right)\right),$$
$$\triangledown_{\kappa+} v\left(x,y\right) = \frac{\sqrt{2}}{2} x^{-p} \cdot \left(\left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z-\right)\right),$$

we obtain

$$\left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z+\right) \geq \left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z-\right), \quad z > 0,$$

which of course means that $u'\left(z-\right) \geq u'\left(z+\right)$ for $z > 0$. Suppose, on the contrary, that for some $z_0 > 0$ we have $\left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z-\right) = 0$. Then, we have that $\triangledown_{\kappa+} v\left(\frac{1}{z_0}, 1\right) = 0$, which, together with the fact that $\xi \to v\left(\frac{1}{z_0} + \xi, 1-\xi\right)$ is concave and nondecreasing for $\xi \in [0,1]$, shows that $v\left(\frac{1}{z_0} + \xi, 1-\xi\right) = v\left(\frac{1}{z_0} + 1, 0\right)$ for $\xi \in [0,1]$. This means that

$$\triangledown_{\kappa} v\left(\frac{1}{z_0} + 1, 0\right) = \frac{\sqrt{2}}{2}\left(\frac{1}{z_0} + 1\right)^{-p} \cdot \left(\left(1-p\right) u\left(0\right) - u'\left(0\right)\right) = 0,$$

which is a contradiction to the boundary condition

$$u'\left(0\right) \leq \frac{\lambda}{1+\lambda}\left(1-p\right) u\left(0\right).$$

Therefore, for any $z > 0$ we have

$$\left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z+\right) \geq \left(1-p\right) u\left(z\right) - \left(z+1\right) u'\left(z-\right) > 0. \tag{37}$$

Assume now that a C^2 function φ touches u from above at some interior point $z \in (0, \infty)$. If $u\left(z\right) = u_s\left(z\right)$ we can use the test function u_s (which is a strict subsolution) for the supersolution u to obtain a contradiction. The contradiction argument works even if $z = (1/2)^{1/\varepsilon}$ is the only exceptional point where u_s is not C^2. Therefore, $u\left(z\right) > u_s\left(z\right)$. From (37) we can easily conclude that

$$\left(1-p\right) \varphi\left(z\right) - \left(z+1\right) \varphi'\left(z\right) > 0.$$

Assume now that u does *not* satisfy the subsolution property, which translates to

$$\sup_{c \geq 0, \pi \in \Delta} \left(\mathcal{L}_{c,\pi} \varphi\right)\left(z\right) < 0. \tag{38}$$

Since $\left(1-p\right) \varphi\left(z\right) - \left(z+1\right) \varphi'\left(z\right) > 0$ we can conclude that

$$\left(1-p\right) \varphi\left(z\right) - z\varphi'\left(z\right) > 0, \tag{39}$$

and

$$-\beta\varphi + r \cdot D_x\left[\varphi\right] + \widetilde{V}\left(D_x\left[\varphi\right]\right) + \sup_{\pi \in \Delta}\left\{\mathcal{B}^T\left[\varphi\right]\pi + \frac{1}{2}\pi^T \mathcal{A}\left[\varphi\right]\pi + \int_{\mathbb{R}^l} \mathbf{R}\varphi(z,\pi,\eta)\mathbf{q}\left(d\eta\right)\right\} < 0 \tag{40}$$

where $D_x\left[\varphi\right] = \left(1-p\right) \varphi\left(z\right) - z\varphi'\left(z\right).$

Because the supremum is taken over a compact set and thus the left-hand side of the above equation is continuous in z, relations (39) or (40) actually hold in a small neighborhood $(z-\delta, z+\delta)$

of z, not just at z. Considering an even smaller δ we have that $u(\omega) < \varphi(\omega)$ for $\omega \in [z-\delta, z+\delta]$ if $\omega \neq z$. Now, for ε small enough, which means at least as small as

$$\varepsilon_0 \triangleq \min_{\frac{\delta}{2} \leq |\omega - z| \leq \delta} (\varphi(\omega) - u(\omega)),$$

but maybe much smaller, we define the function

$$\widetilde{u}(\omega) \triangleq \begin{cases} \min\{u(\omega), \varphi(\omega) - \varepsilon\}, & \omega \in [z-\delta, z+\delta], \\ u(\omega), & \omega \notin [z-\delta, z+\delta]. \end{cases}$$

If ε is small enough, we have that $\widetilde{u} \in \mathcal{S}$ and $\widetilde{u}$ is strictly smaller than u (around z), and this is a contradiction.

6. From above, we already know that

$$u'(0) \leq \frac{\lambda}{1+\lambda}(1-p)\, u(0).$$

Let us now prove the above inequality is actually an equality. Assume now that the inequality above is strict. Since $u_s'(0) = a > \frac{\lambda}{1+\lambda}(1-p)\, u_s(0)$ and $u \geq u_s$, this rules out the possibility that $u(0) = u_s(0)$, so we have $u(0) > u_s(0)$. Also because $u_s(0) = w_* - \xi$ for arbitrarily small $\xi > 0$, we have $u(0) > w_*$. This implies

$$-\beta u(0) + r(1-p)\, u(0) + \widetilde{V}((1-p)\, u(0)) < 0.$$

Recall that for fixed x, the function $y \to x^{1-p} u(y/x)$ is concave, this means u is concave and therefore two times differentiable on a dense set of $(0,\infty)$. Then, we can find $z_0 \in (0,\infty)$ very close to 0 such that

$$-\beta u(z_0) + r \cdot D_x[u(z_0)] + \widetilde{V}(D_x[u(z_0)]) < 0$$

and $u(z)$ solves the HJB equation (30) at $z = z_0$ in the classical sense. More precisely, we have

$$\begin{aligned} -\beta u(z_0) + r \cdot D_x[u(z_0)] &+ \widetilde{V}(D_x[u(z_0)]) \\ &+ \sup_{\pi \in \Delta} \left\{ \mathcal{B}^T[u(z_0)]\, \pi + \frac{1}{2}\pi^T \mathcal{A}[u(z_0)]\, \pi + \int_{\mathbb{R}^l} \mathbf{R}u(z_0, \pi, \eta)\mathbf{q}(d\eta) \right\} = 0. \end{aligned}$$

and the supremum part of the above equation is strictly positive, which means $\widehat{\pi}(z_0) \neq \mathbf{0}$. With $u'(z_0)$ being very close to $u'(0)$, we can find a number a' such that

$$u'(z_0) \leq u'(0) < a' < \frac{\lambda}{1+\lambda}(1-p)\, u(0).$$

and without loss of generality, we also choose a' to be very close to $u'(z_0)$.

Moreover, we observe that the left-hand side of the above equation is continuous in both u' and u'', and increasing in u'' given $\widehat{\pi}(z_0) \neq \mathbf{0}$, since each element of $\mathcal{A}[u]$ is increasing in u''. This allows us to choose a (possibly very large) $b > 0$ together with a' above such that the function

$$\psi(z) = u(0) + a'z - \frac{1}{2}bz^2$$

is a classical strict supersolution at $z = z_0$. More precisely, we have

$$\begin{aligned} -\beta \psi(z_0) + r \cdot D_x[\psi(z_0)] &+ \widetilde{V}(D_x[\psi(z_0)]) \\ &+ \sup_{\pi \in \Delta} \left\{ \mathcal{B}^T[\psi(z_0)]\, \pi + \frac{1}{2}\pi^T \mathcal{A}[\psi(z_0)]\, \pi + \int_{\mathbb{R}^l} \mathbf{R}\psi(z_0, \pi, \eta)\mathbf{q}(d\eta) \right\} < 0. \end{aligned}$$

Then, continuity would imply that $\psi(z)$ is actually a classical strict supersolution in a small neighborhood $(0,\delta)$ of $z=0$. In addition, it satisfies $\psi'(0)=a'<\frac{\lambda}{1+\lambda}(1-p)\psi(0)$. Thus, if δ is small enough, we have that $u(z)<\psi(z)$ on $(0,\delta)$, and

$$(1-p)\psi(z)-z\psi'(z)>0,\ z\in[0,\delta].$$

Now, for a very small ε, at least as small as

$$\varepsilon_0\triangleq\min_{z\in\left[1+\frac{\delta}{2},1+\delta\right]}(\psi(z)-u(z))$$

but possibly even smaller, we have that the function

$$\widetilde{u}(z)\triangleq\begin{cases}\min\{u(z),\psi(z)-\varepsilon\},\ z\in[0,\delta],\\ u(z),\ z\in[\delta,\infty),\end{cases}$$

is actually an element of $\mathcal{S}$, contradicting with the assumption that u is the infimum over $\mathcal{S}$.

□

3.5 Smoothness of the viscosity solution

Theorem 3.8. *The function u in Theorem 3.7 is C^2 on $[0,\infty)$ and satisfies the conditions*

$$(1-p)u(z)-zu'(z)>0,\quad z\geq 0.$$

Moreover, it is a solution of the equation

$$\begin{cases}\sup_{c>0,\pi\in\Delta}\mathcal{L}_{c,\pi}u=-\beta u+r\cdot(D_x[u])+\widetilde{V}(D_x[u])\\ +\sup_{\pi\in\Delta}\left\{\mathcal{B}^T[u]\pi+\frac{1}{2}\pi^T\mathcal{A}[u]\pi+\int_{\mathbb{R}^l}\mathbf{R}(z,\pi,\eta)\mathbf{q}(d\eta)\right\}=0,\ z>0,\\ -\lambda(1-p)u(0)+(1+\lambda)u'(0)=0.\end{cases}$$

Proof. First, we point out that the dual function $\widetilde{V}(y)$ is defined for all values of y, not only $y>0$. More precisely,

$$\widetilde{V}(y)=\begin{cases}\frac{p}{1-p}y^{\frac{p-1}{p}},\ y>0,\\ +\infty,\ y\leq 0\end{cases}\text{ for } p<1\ ,\ \widetilde{V}(y)=\begin{cases}\frac{p}{1-p}y^{\frac{p-1}{p}},\ y\geq 0,\\ +\infty,\ y<0\end{cases}\text{ for } p>1\ .$$

Let $z_0>0$ such that $u'(z_0-)>u'(z_0+)$. For each $u'(z_0+)<a<u'(z_0-)$ and $b>0$ very large we use the function

$$\psi(z)\triangleq u(z_0)+a(z-z_0)-\frac{1}{2}b(z-z_0)^2$$

as a test function at $z=z_0$ for the viscosity subsolution property, so

$$\sup_{c\geq 0,\pi\in\Delta}\mathcal{L}_{c,\pi}\psi\geq 0.$$

Since $(1-p)u(z_0)-z_0a>(1-p)u(z_0)-z_0u'(z_0-)>0$ the above equation can be rewritten as

$$\begin{aligned}-\beta u(z_0)&+r((1-p)u(z_0)-z_0a)+\widetilde{V}((1-p)u(z_0)-z_0a)\\ &+\sup_{\pi\in\Delta}\left\{\mathcal{B}^T[\psi(z_0)]\pi+\frac{1}{2}\pi^T\mathcal{A}[\psi(z_0)]\pi+\int_{\mathbb{R}^l}\mathbf{R}\psi(z_0,\pi,\eta)\mathbf{q}(d\eta)\right\}\geq 0.\end{aligned}$$

We note that the above inequality holds even when $b\to\infty$ and the left-hand side is decreasing in b given $\widehat{\pi}(z_0)\neq\mathbf{0}$, hence we must have $\widehat{\pi}(z_0)=\mathbf{0}$. This implies

$$-\beta u(z_0)+r((1-p)u(z_0)-z_0a)+\widetilde{V}((1-p)u(z_0)-z_0a)\geq 0,\quad a\in\left(u'(z_0+),u'(z_0-)\right).$$

It is easy to see that the function

$$g(a) := -\beta u(z_0) + r((1-p)u(z_0) - z_0 a) + \widetilde{V}((1-p)u(z_0) - z_0 a)$$

is not flat on any nontrivial interval within its domain. We must have

$$-\beta u(z_0) + r((1-p)u(z_0) - z_0 a) + \widetilde{V}((1-p)u(z_0) - z_0 a) > 0 \text{ for some } a \in \left(u'(z_0+), u'(z_0-)\right).$$

and we can also assume, without loss of generality, that a is very close to $u'(z_0-)$. So

$$-\beta u(z_0) + r\left((1-p)u(z_0) - z_0 u'(z_0-)\right) + \widetilde{V}\left((1-p)u(z_0) - z_0 u'(z_0-)\right) > 0.$$

Since $u'(z-)$ is left continuous, and the function u is two times differentiable on a dense set $\mathcal{D} \subset (0,\infty)$ by convexity, there exists $z > 0$ very close to z_0 such that $z \in \mathcal{D}$, and

$$-\beta u(z) + r\left((1-p)u(z) - zu'(z)\right) + \widetilde{V}\left((1-p)u(z) - zu'(z)\right) > 0.$$

However, this would contradict with the viscosity supersolution property at z, which reads

$$\begin{aligned}-\beta u(z) + &r\left((1-p)u(z) - zu'(z)\right) + \widetilde{V}\left((1-p)u(z) - zu'(z)\right) \\ &+ \sup_{\pi\in\Delta}\left\{\mathcal{B}^T[u(z)]\pi + \frac{1}{2}\pi^T\mathcal{A}[u(z)]\pi + \int_{\mathbb{R}^l}\mathbf{R}\psi(z,\pi,\eta)\mathbf{q}(d\eta)\right\} \le 0,\end{aligned}$$

since the supremum part of the left-hand side above is always non-negative. We obtained a contradiction, so we have proved that

$$u'(z_0-) = u'(z_0+) \quad \forall\, z_0 > 0.$$

In other words, u' is well defined and continuous on $[0,\infty)$. In addition, $(1-p)u(z) - zu'(z) > 0$ for $z \ge 0$. Applying again the viscosity solution property at a point where u is two times differentiable we obtain

$$-\beta u(z) + r\left((1-p)u(z) - zu'(z)\right) + \widetilde{V}\left((1-p)u(z) - zu'(z)\right) \le 0, \quad z \in \mathcal{D}.$$

Using continuity and the density of $\mathcal{D}$, we get

$$f(z) \triangleq \beta u(z) - r\left((1-p)u(z) - zu'(z)\right) - \widetilde{V}\left((1-p)u(z) - zu'(z)\right) \ge 0, \quad z \ge 0. \qquad (41)$$

The function f defined in (41) is continuous. Also, $u(0) > w_*$ as seen in Theorem 3.2, and u is nondecreasing since $y \to x^{1-p}u(y/x)$ is nondecreasing in $y \in [0,\infty)$, it follows that $u(z) > w_*$ for all $z \ge 0$. Hence, $\widehat{\pi} \ne \mathbf{0}$ on $[a,b]$ for any open interval $(a,b) \subset [0,\infty)$. Therefore, $f(z) > 0$ on $[a,b]$ due to the HJB equation. Now, rewrite the HJB equation (30) in the following form,

$$H\left(z, u''\right) = 0$$

where H is continuous and strictly increasing in its second variable. Note that H depends on its first variable z through $u(z)$ and $u'(z)$, which are continuous. This implies that the HJB equation can further be rewritten as

$$u'' = h(z)$$

where h is continuous. Now, u is a viscosity solution of the equation

$$u - u'' = u - h(z), \ z \in (a,b) \subset [0,\infty),$$

and the right-hand side is continuous in z on $[a,b]$. Comparing to the classical solution of this equation with the very same right-hand side and Dirichlet boundary conditions at a and b, we get that u is C^2 on $[a,b]$. We point out that the comparison argument between the viscosity solution and the classical solution is straightforward and does not involve any doubling argument.

Therefore u is C^2 on $(1,\infty)$ and satisfies the HJB equation. Since $u(1) > w_*$ which reads $f(1) > 0$, for f defined in (41), we can then use continuity and pass to the limit in the HJB equation for $z \searrow 1$ to conclude that u is C^2 in $[1,\infty)$ and the HJB equation is satisfied at the boundary as well. $\square$

Lemma 3.9. *The function u is strictly increasing on $[0,\infty)$ and*

$$\lim_{z\to\infty} u(z) = u_0, \quad \lim_{z\to\infty} zu'(z) = 0, \quad \lim_{z\to\infty} z^2u''(z) = 0.$$

Proof. Recall that $y \to x^{1-p}u(y/x)$ is concave and nondecreasing in $y \in [0,\infty)$, this means u is concave and nondecreasing. Since u is nondecreasing and bounded, there exists

$$u(\infty) \triangleq \lim_{z\to\infty} u(z) \in (-\infty,\infty).$$

Now, since u is bounded and u' is continuous we conclude, by contradiction, that there exists a sequence $z_n \nearrow \infty$ such that

$$z_n u'(z_n) \to 0, \ n \to \infty.$$

(Otherwise we would have $zu'(z) \geq \varepsilon$ for some ε for large z, which contradicts boundedness.) We let

$$0 \geq A := \liminf_{z\to\infty} zu'(z) \leq \limsup_{z\to\infty} zu'(z) =: B \geq 0.$$

For fixed $C \in \mathbb{R}$, denote by

$$f_C(z) = Cu + zu', \ z \geq 1.$$

The function f_C is continuous and

$$\liminf_{z\to\infty} f_C(z) = Cu(\infty) + A \leq Cu(\infty) + B = \limsup_{z\to\infty} f_C(z).$$

Assume, on the contrary, that $0 < B \leq \infty$. Since $\lim_{n\to\infty} f_C(z_n) = Cu(\infty) < Cu(\infty) + B$, we can choose the points $\eta_n \in (z_n, z_{n+1})$ (interior points, and eventually for a subsequence n_k rather than for each n) for which f_C attains the maximum on $[z_n, z_{n+1}]$ such that $f_C(\eta_n) \to Cu(\infty) + B$, which is the same as $\eta_n u'(\eta_n) \to B$. Since f_C attains the interior maximum on each interval at η_n, we have $f_C'(\eta_n) = (1+C)u'(\eta_n) + \eta_n u''(\eta_n) = 0$. Recall that $x \to x^{1-p}u(y/x)$ is concave in $x \in (0,n]$, which implies that

$$-p(1-p)u(\eta_n) + 2p\eta_n u'(\eta_n) + \eta_n^2 u''(\eta_n) \leq 0,$$

or

$$-p(1-p)u(\eta_n) + (2p-1-C)\eta_n u'(\eta_n) \leq 0.$$

Passing to the limit, we obtain that

$$-p(1-p)u(\infty) + (2p-1-C)B \leq 0$$

for each $C \in \mathbb{R}$, which means that $B = 0$. Similarly, we obtain $A = 0$ so $zu'(z) \to 0$. Now, since zu' is bounded and $(zu')'$ is continuous we conclude, by contradiction, that there exists a sequence $z_n \nearrow \infty$ such that

$$(z_n)^2 u''(z_n) \to 0, \ n \to \infty.$$

Passing to the limit along z_n's in the HJB equation, we obtain

$$\begin{aligned} &-\beta u(\infty) + r(1-p)u(\infty) + \widetilde{V}((1-p)u(\infty)) \\ &+ \sup_{\pi\in\Delta} \left\{ \begin{array}{c} (1-p)u(\infty)\alpha^T\pi + \frac{1}{2}(-p(1-p)u(\infty))\pi^T\sigma\sigma^T\pi \\ + \int_{\mathbb{R}^l} \left\{ \left(\left(1+\pi^T\mathbf{J}(\eta)\right)^{1-p} - 1 \right) u(\infty) \right\} \mathbf{q}(d\eta) \end{array} \right\} = 0. \end{aligned} \tag{42}$$

As already pointed out, the above equation has a unique solution $u(\infty)$ in $[w_*, u_0]$, namely, $u(\infty) = u_0$ so $u(z) \to u_0$ as $z \to \infty$. Going back to the ODE for all $z \to \infty$ and not only along the subsequence, we obtain $z^2u''(z) \to 0$ as well.

Now we show that u is strictly increasing. Suppose otherwise, since u is nondecreasing and concave, it is only possible that $u(z) = u(\infty)$ for $z \geq z_0$ for some $z_0 > 0$. Plugging $u(z_0) = u(\infty)$ into the HJB equation we have

$$-\beta u(\infty) + r(1-p)u(\infty) + \widetilde{V}((1-p)u(\infty)) + \sup_{\pi\in\Delta}\left\{\begin{array}{l}(1-p)u(\infty)\alpha^T\pi + \frac{1}{2}(-p(1-p)u(\infty))\pi^T\sigma\sigma^T\pi \\ + \int_{\mathbb{R}^l}\left\{\left(\left(\begin{array}{c}1+\pi^T\mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda}\left[\pi^F\mathbf{J}^E(\eta)-z_0\right]^+\end{array}\right)^{1-p} - 1\right)\cdot u(\infty)\right\}\mathbf{q}(d\eta)\end{array}\right\} = 0,$$

which is a contradiction with (42). Therefore, u is strictly increasing. □

3.6 Optimal policies and verification

Proposition 3.10. *Let $\theta(x,y)$ and $\gamma(x,y)$ be two Lipschitz functions in both arguments on the two-dimensional domain D. The closed-loop equation obtained from (10) by (formally) feeding back the controls*

$$\theta_s = \theta(X_{s-}, Y_{s-}), \;\; \gamma_s = \gamma(X_{s-}, Y_{s-}),$$

has a unique strong solution (X,Y).

Proof. Consider the operator

$$(N, L) \to (X, Y)$$

defined by

$$\begin{cases} X_t: &= x + \int_0^t rX_s ds + \int_0^t \theta^F(N_{s-}, L_{s-})\left(\frac{dF_s}{F_{s-}} - rds\right) \\ &+ \int_0^t \sum_{i=1}^n \theta^i(N_{s-}, L_{s-})\left(\frac{dS_s^i}{S_{s-}^i} - rds\right) - \int_0^t \gamma(N_{s-}, L_{s-})ds - \lambda M_t, \\ Y_t: &= y - \int_0^t \theta^F(N_{s-}, L_{s-})\left(\frac{dF_s}{F_{s-}} - \frac{dB_s}{B_{s-}}\right) + (1+\lambda)M_t, \\ & \int_0^t \mathbf{1}_{\{Y_s>0\}} dM_s = 0. \end{cases}$$

In words, we obtain (X,Y) from (N,L) by solving the state equation (10) for $\theta_s = \theta(N_{s-}, L_{s-})$ and $\gamma_s = \gamma(N_{s-}, L_s)$. According to Proposition 2.2, the solution (X,Y) is given by

$$\begin{aligned} X_t &= \\ &= e^{rt}\left\{x + \int_0^t e^{-rs}\theta^T(N_{s-}, L_{s-})\left(\alpha ds + \sigma d\mathbf{W}_s + \int_{\mathbb{R}^l}\mathbf{J}(\eta)\mathcal{N}(d\eta, ds)\right) - \int_0^t e^{-rs}\gamma(N_{s-}, L_s)ds\right\} \\ &- e^{rt}\left\{\frac{\lambda}{1+\lambda}\int_0^t e^{-rs}d\left(\sup_{0\leq s\leq t}\left[\int_0^s \theta^F(N_{u-}, L_{u-})\left(\mu^E du + \sigma^E d\mathbf{W}_u + \int_{\mathbb{R}^l}\mathbf{J}^E(\eta)\mathcal{N}(d\eta, du)\right) - y\right]^+\right)\right\} \end{aligned}$$

and

$$\begin{aligned} Y_t = y &- \int_0^t \theta^F(N_{s-}, L_{s-})\left(\mu^E ds + \sigma^E d\mathbf{W}_s + \int_{\mathbb{R}^l}\mathbf{J}^E(\eta)\mathcal{N}(d\eta, ds)\right) \\ &+ \sup_{0\leq s\leq t}\left[\int_0^s \theta^F(N_{u-}, L_{u-})\left(\mu^E du + \sigma^E d\mathbf{W}_u + \int_{\mathbb{R}^l}\mathbf{J}^E(\eta)\mathcal{N}(d\eta, du)\right) - y\right]^+. \end{aligned}$$

Now we can use the usual estimates in the Ito's theory of SDEs to obtain

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left\|\left(X_s^1 - X_s^2, Y_s^1 - Y_s^2\right)\right\|^2\right] \leq C^*(T)\int_0^t \mathbb{E}\left[\left\|\left(N_s^1 - N_s^2, L_s^1 - L_s^2\right)\right\|^2\right]ds,$$

as long as $0 \leq t \leq T$ for each fixed $T > 0$, where $C^*(T) < \infty$ is a constant depending on the Lipschitz constants of θ and γ, and quantity $\int_{\mathbb{R}^l}|\mathbf{J}(\eta)|_2^2\mathbf{q}(d\eta) < \infty$ by assumption, as well as the time horizon T. This allows us to prove pathwise uniqueness using Grownwall's inequality and also to prove existence using a Picard iteration. □

Proof of Proposition 3.3. From Proposition 3.5 we can see that

$$\widehat{\theta}(x,y) \triangleq \begin{cases} x\widehat{\pi}(x,y), \; x>0, y\geq 0, \\ 0, \; x\leq 0, y\geq 0, \end{cases}$$

and

$$\widehat{\gamma}(x,y) \triangleq \begin{cases} x\widehat{c}(x,y), \; x>0, y\geq 0, \\ 0, \; x\leq 0, y\geq 0, \end{cases}$$

are globally Lipschitz in the domain $x \in \mathbb{R}, y \geq 0$. Therefore, according to Proposition 3.10 the equation has a unique solution $\left(\widehat{X}, \widehat{Y}\right) \in \mathbb{R}\times [0,\infty)$. It only remains to prove that $\widehat{X} > 0$ in order to finish the proof of Proposition 3.3, and this is shown in the next proposition. □

Proposition 3.11. *Let $x>0, y\geq 0$. Assume that the predictable process (π, c) satisfies the integrability condition is in Proposition 2.2. If (X,Y) is a solution to the equation*

$$\begin{cases} X_t &= x + \int_0^t rX_s ds + \int_0^t \pi^F X_{s-}\left(\frac{dF_s}{F_{s-}} - rds\right) + \int_0^t \sum_{i=1}^n \pi^i X_{s-}\left(\frac{dS_s^i}{S_{s-}^i} - rds\right) - \int_0^t cX_{s-}ds - \lambda M_t, \\ Y_t &= y - \int_0^t \pi^F X_{s-}\left(\frac{dF_s}{F_{s-}} - \frac{dB_s}{B_{s-}}\right) + (1+\lambda) M_t, \\ & \int_0^t \mathbf{1}_{\{Y_s>0\}} dM_s = 0. \end{cases}$$

then

$$X_t > 0, Y_t \geq 0, \; 0 \leq t < \infty.$$

Proof. Denote by $\tau \triangleq \{t\geq 0 : X_t = 0\}$. We can apply Ito's formula to $N_t = \log(X_t)$ and take into account that $Y_t \geq 0$ (also $Y_{t-}\geq 0$) and $Y_t = Y_{t-} = 0$ on the support of dM^c to obtain

$$\begin{aligned} N_t &= \log x + R_t - \lambda \int_0^t \frac{dM_s^c}{X_{s-}} \\ &+ \int_0^t \int_{\mathbb{R}^l} \log\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)\right) \mathcal{N}(ds, d\eta) \\ &= \log x + R_t - \frac{\lambda}{1+\lambda}\int_0^t \frac{1}{X_{s-}} d\left(\sup_{0\leq u\leq s}\left[\int_0^u \theta_\tau^F\left(\mu^E d\tau + \sigma^E d\mathbf{W}_\tau\right) - y\right]^+\right) \\ &+ \int_0^t \int_{\mathbb{R}^l} \log\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)\right) \mathcal{N}(ds, d\eta) \\ &= \log x + R_t - \frac{\lambda}{1+\lambda}\int_0^t \pi_s^F 1_{\{dM^c>0\}}\left(\mu^E ds + \sigma^E d\mathbf{W}_s\right) \\ &+ \int_0^t \int_{\mathbb{R}^l} \log\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)\right) \mathcal{N}(ds, d\eta) \end{aligned}$$

where

$$R_t \triangleq \int_0^t \left(r + \pi_s^T\alpha - c_s - \frac{1}{2}\pi_s^T\sigma^T\sigma\pi_s\right) ds + \int_0^t \pi_s^T \sigma d\mathbf{W}_s, \; t\geq 0.$$

We observe that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^l} \log\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left(\left[\pi^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)\right) \mathcal{N}(ds, d\eta) \\ &\geq \int_0^t \int_{\mathbb{R}^l} \log\left(1 + \pi_s^T \mathbf{J}(\eta)\right) \mathcal{N}(ds, d\eta) \\ &> \int_0^t \int_{\mathbb{R}^l} \pi_s^T \mathbf{J}(\eta) \mathcal{N}(ds, d\eta_i) - \int_0^t \int_{\mathbb{R}^l} \pi_s^T \mathbf{J}(\eta)^2 \pi_s \mathcal{N}(ds, d\eta_i) \\ &> -\infty, \end{aligned}$$

according to the assumption about jumps in (4). And because

$$\lim_{t\nearrow\tau} R_t > -\infty \quad \text{on} \quad \{\tau < \infty\},$$

we can then obtain that $\tau = \infty$. □

Proof of Theorem 3.4. First we verify that the condition in (16) is satisfied. Using the notations (15) and (21), together with $v(x,y) = x^{1-p}u(y/x)$, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^l} e^{-\beta s} \left|\mathbf{U}(X_{s-}, Y_{s-}, \theta_s, \eta)\right| \mathbf{q}(d\eta)\, ds \\
&= \int_0^t \int_{\mathbb{R}^l} e^{-\beta s} (X_{s-})^{1-p} \left|\mathbf{R}\left(\frac{Y_{s-}}{X_{s-}}, \pi_s, \eta\right)\right| \mathbf{q}(d\eta)\, ds \\
&\le \max_{z\ge 0} |u(z)| \max_{\pi\in\Delta} \int_{\mathbb{R}^l} \left|\left(1+\pi^T \mathbf{J}(\eta)\right)^{1-p} - 1\right| \mathbf{q}(d\eta) \cdot \int_0^t e^{-\beta s} (X_{s-})^{1-p}\, ds \\
&+ \max_{z\ge 0} |u'(z)| \max_{\pi\in\Delta} \int_{\mathbb{R}^l} \left|\left(1+\pi^T \mathbf{J}(\eta)\right)^{-p} \pi^T \mathbf{J}(\eta)\right| \mathbf{q}(d\eta) \cdot \int_0^t e^{-\beta s} (X_{s-})^{1-p}\, ds \\
&+ \max_{z\ge 0} |u'(z)| \max_{\pi\in\Delta} \int_{\mathbb{R}^l} \left|\left(1+\pi^T \mathbf{J}(\eta)\right)^{-p} \pi^T \mathbf{J}(\eta)\right| \mathbf{q}(d\eta) \cdot \int_0^t e^{-\beta s} (X_{s-})^{-p} Y_{s-} ds \\
&< \infty \text{ a.s.}
\end{aligned}$$

The last inequality follows from the assumption (14) and u, u' are bounded, and that X_{s-}, Y_{s-} are left continuous with right limits. The second to last inequality holds true since, from the definition of the operator $\mathbf{R}$ in (21) and applying the Intermediate Value Theorem we have, for some intermediate (random) value ξ:

$$\begin{aligned}
&\mathbf{R}\left(\frac{Y_{s-}}{X_{s-}}, \pi_s, \eta\right) = \\
&= \left(\left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)^{1-p} - 1\right) \cdot u\left(\frac{Y_{s-}}{X_{s-}}\right) \\
&- \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)^{-p} \cdot \left(\pi_s^F \mathbf{J}^E(\eta) - \left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right) u'(\xi) \\
&- \left(1 + \pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right)^{-p} \cdot \left(\pi_s^T \mathbf{J}(\eta) - \frac{\lambda}{1+\lambda}\left[\pi_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}}\right]^+\right) \frac{Y_{s-}}{X_{s-}} u'(\xi).
\end{aligned}$$

According to Lemma 3.1, the process

$$V_t = \int_0^t e^{-\beta s} U(\gamma_s)\, ds + e^{-\beta t} v(X_t, Y_t),\ t \ge 0,$$

is a local supermartingle for each admissible control and a local martingale for the feedback control $\left(\widehat{\theta}, \widehat{\gamma}\right)$.

1. If $p > 1$, then for a sequence of stopping times τ_k we have

$$v(x,y) = \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U(\widehat{\gamma}_s)\, ds + e^{-\beta\tau_k} v\left(\widehat{X}_{\tau_k}, \widehat{Y}_{\tau_k}\right)\right] \le \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U(\widehat{\gamma}_s)\, ds\right].$$

   Letting $k \to \infty$ and using monotone convergence theorem, we get

$$v(x,y) \le \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} U(\widehat{\gamma}_s)\, ds\right].$$

Now, let $(\theta,\gamma) \in \mathcal{A}(x,y)$ be admissible controls. It is easy to see from Proposition 2.2 that $(\theta,\gamma) \in \mathcal{A}(x+\varepsilon,y)$, and the wealth X corresponding to (θ,γ) starting at $x+\varepsilon$ with high-watermark y satisfies $X > \varepsilon$. Using the local supermaringale property along the solution (X,Y) starting at $(x+\varepsilon,y)$ with controls (θ,γ), we obtain

$$v(x+\varepsilon,y) \geq \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U(\gamma_s)\,ds + e^{-\beta\tau_k} v(X_{\tau_k},Y_{\tau_k})\right].$$

However, since $X > \varepsilon$ we obtain

$$|v(X,Y)| \leq C\varepsilon^{1-p},$$

where C is a bound on $|u|$. Therefore, we can again let $k \to \infty$ and use monotone convergence theorem together with the bounded convergence theorem (respectively for the two terms on the right-hand side) to obtain

$$v(x+\varepsilon,y) \geq \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} U(\gamma_s)\,ds\right]$$

for all $(\theta,\gamma) \in \mathcal{A}(x,y)$. This means that

$$v(x+\varepsilon,y) \geq \sup_{(\theta,\gamma)\in\mathcal{A}(x,y)} \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} U(\gamma_s)\,ds\right] = V(x,y)$$

and the conclusion follows from letting $\varepsilon \searrow 0$.

2. Let $p < 1$. Then by the local supermartingale property we obtain

$$v(x,y) \geq \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U(\gamma_s)\,ds + e^{-\beta\tau_k} v(X_{\tau_k},Y_{\tau_k})\right] \geq \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U(\gamma_s)\,ds\right].$$

Letting $k \to \infty$ we get

$$v(x,y) \geq \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} U(\gamma_s)\,ds\right]$$

for each $(\theta,\gamma) \in \mathcal{A}(x,y)$.

Now, for the optimal $(\widehat{\pi},\widehat{c})$ (in *proportion* form) we have

$$v(x,y) = \mathbb{E}\left[\int_0^{\tau_k} e^{-\beta s} U\left(\widehat{c}_s\widehat{X}_s\right)ds + e^{-\beta\tau_k} v\left(\widehat{X}_{\tau_k},\widehat{Y}_{\tau_k}\right)\right].$$

If we can show that

$$\mathbb{E}\left[e^{-\beta\tau_k} v\left(\widehat{X}_{\tau_k},\widehat{Y}_{\tau_k}\right)\right] \to 0, \tag{43}$$

then we use monotone convergence theorem to obtain

$$v(x,y) = \mathbb{E}\left[\int_0^{\infty} e^{-\beta s} U\left(\widehat{c}_s\widehat{X}_s\right)ds\right]$$

and finish the proof. Let us now prove (43). The value function $v_0(x,y) \triangleq u_0 x^{1-p}$ corresponding to $\lambda = 0$ is a supersolution of the HJB equation since the constant function u_0 is a supersolution to (36). Using Lemma 3.1 for the function v_0 and denoting by

$$Z_t := e^{-\beta t} v_0\left(\widehat{X}_t,\widehat{Y}_t\right) = u_0 e^{-\beta t}\left(\widehat{X}_t\right)^{1-p},$$

we obtain

$$
\begin{aligned}
&Z_t + \int_0^t e^{-\beta s} \frac{\left(\widehat{c}\widehat{X}_t\right)^{1-p}}{1-p} ds = Z_t + \int_0^t Z_s \frac{(\widehat{c})^{1-p}}{(1-p)\,u_0} ds \\
&\leq \int_0^t (1-p)\left(\widehat{\pi}_s\right)^T \sigma Z_s d\mathbf{W}_s \\
&+ \int_0^t Z_{s-} \int_{\mathbb{R}^l} \left\{ \left(\begin{array}{c} 1 + \widehat{\pi}_s^T \mathbf{J}(\eta) \\ -\frac{\lambda}{1+\lambda} \left[\widehat{\pi}_s^F \mathbf{J}^E(\eta) - \frac{Y_{s-}}{X_{s-}} \right]^+ \end{array} \right)^{1-p} - 1 \right\} \widetilde{\mathcal{N}}(d\eta, ds) \\
&\leq \int_0^t (1-p)\left(\widehat{\pi}_s\right)^T \sigma Z_s d\mathbf{W}_s + \int_0^t Z_{s-} \int_{\mathbb{R}^l} \left\{ \left(1 + \widehat{\pi}_s^T \mathbf{J}(\eta)\right)^{1-p} - 1 \right\} \widetilde{\mathcal{N}}(d\eta, ds).
\end{aligned}
$$

Recall that from Proposition 3.5 we have that $\widehat{c} \geq c_0$. This means that if we denote by $\delta := \frac{c_0^{1-p}}{(1-p)u_0} > 0$, then we have

$$
\begin{aligned}
&Z_t + \int_0^t \delta Z_s ds \\
&\leq \int_0^t (1-p)\left(\widehat{\pi}_s\right)^T \sigma Z_s d\mathbf{W}_s + \int_0^t Z_{s-} \int_{\mathbb{R}^l} \left\{ \left(1 + \widehat{\pi}_s^T \mathbf{J}(\eta)\right)^{1-p} - 1 \right\} \widetilde{\mathcal{N}}(d\eta, ds).
\end{aligned}
$$

Using the exponential solution for the equation obtained by using the equality sign in the differential (formal) inequality above, and the comparison principle, together with (14) we can obtain that $\{Z_t\}_{t\geq 0}$ is uniformly integrable. Now taking into account that $e^{-\beta t} v(X_t, Y_t) \leq Z_t \to 0$ a.s. for $t \to \infty$, we obtain (43) and the proof is complete.

□

4 Quantitative analysis

4.1 Certainty equivalent

A useful method of evaluating the impact of the parameter λ, namely the proportional high-water mark fee, is to compute the so-called certainty equivalent wealth. By definition, the certainty equivalent wealth is such a size of initial bankroll $\widetilde{x}$ that the agent would be indifferent between $\widetilde{x}$ when paying zero fees and wealth x when paying proportional high-water mark fees λ, all other parameters being the same.

By equating $v_0(\widetilde{x}) = \widetilde{x}^{1-p} u_0$ and $v(x, y) = x^{1-p} u(z)$, we solve for quantity

$$
\frac{\widetilde{x}(z)}{x} = \left(\frac{u(z)}{u_0}\right)^{\frac{1}{1-p}} = \left((1-p)\, c_0^p u(z)\right)^{\frac{1}{1-p}}, \quad z \geq 0,
$$

which is the relative amount of wealth needed to achieve the same utility if no fee is paid (which also quantifies the proportional loss of wealth).

Another method is to find the certainty equivalent excess return $\widetilde{\alpha} < \alpha$ so that the value function obtained by using $\widetilde{\alpha}$ and no fee is equal to the value function when the return is α but the high-water mark performance fee is paid.

Keeping all other parameters the same, the value function for zero high-water mark performance fee corresponding to $\widetilde{\alpha}$ is given by

$$
\widetilde{u}_0(\widetilde{\alpha}) = \frac{1}{1-p} \widetilde{c}_0(\widetilde{\alpha})^{-p}, \quad z \geq 0.
$$

where $\widetilde{c}_0$ is defined as in (27) with α being replaced by $\widetilde{\alpha}$.

Therefore, we are looking for the solution to the equation

$$\widetilde{u}_0\left(\widetilde{\alpha}\left(z\right)\right) = u\left(z\right),$$

In general, this equation above is difficult to solve analytically. However, in the particular case where the jump term vanishes, i.e., $\mathbf{q} = \mathbf{0}$, then

$$\widetilde{c}_0\left(\widetilde{\alpha}\right) \triangleq \frac{\beta}{p} - r\frac{1-p}{p} - \frac{1}{2}\frac{1-p}{p^2} \cdot \widetilde{\alpha}^T \left(\sigma\sigma^T\right)^{-1} \widetilde{\alpha}.$$

and $\widetilde{\alpha}$ is implicitly given by

$$\widetilde{\alpha}^T \left(\sigma\sigma^T\right)^{-1} \widetilde{\alpha} = \frac{2p^2}{1-p}\left(\frac{\beta}{p} - r\frac{1-p}{p} - \left(\left(1-p\right)u\left(z\right)\right)^{-\frac{1}{p}}\right), \quad z \geq 0.$$

Because $\widetilde{\alpha}$ and α differ only in their first element ($\widetilde{\alpha}_F$ and α_F, respectively), the above is a quadratic equation of $\widetilde{\alpha}_F$. Now, the relative certainty equivalent excess return would be $\widetilde{\alpha}_F/\alpha_F$, which also equals to the relative certainty equivalent Sharpe ratio due to constant σ_F.

4.2 Numerical examples

To the best of our knowledge, there is no closed-form solution for our optimization problem at hand. In order to understand the impact of the high-water mark fees on the investor, we need to resort to numerics. The paper [18] gave numerical results for the case in which there is only a single risky asset, the hedge fund, the interest rate is zero and the fund share price is a continuous process. Specifically, the authors numerically solve the HJB equation for the value function using an iterative method, then use the results to describe the optimal investment/consumption proportions, as well as the certainty equivalent wealth and the certainty equivalent $\widetilde{\alpha}$ (which we defined in the last section). Our numeric experiment generalizes the result of [18] in two ways:

1. In addition to a hedge fund F, we introduce another stock S, possibly correlated with F, and investigate the value function, the optimal investment and consumption proportions, as well as the certainty equivalent wealth and the certainty equivalent $\widetilde{\alpha}$ in this multiple-asset case.

2. On top of the multiple-asset case described above, we incorporate jumps into the processes of F and S, and study the effect of jumps by comparison.

We follow [18] and set our *benchmark parameters* as follows,

$$\begin{gathered} p_0 = 7,\ \beta_0 = 5\%,\ \mu_0^F = 20\%,\ \mu_0^S = 10\%,\ r_0 = 4\%, \\ \sigma_0^F = 20\%,\ \sigma_0^S = 20\%,\ \rho_0 = 0,\ \lambda_0 = 25,\ \mathbf{q} = \mathbf{0}. \end{gathered}$$

The Merton values for these parameters are:

$$\pi_0^F = 0.571, \pi_0^S = 0.214, c_0 = 0.0861$$

Note that for all graphs below the horizontal axis is the variable z, the "relative distance to paying HWM fees". We remind the reader that the values for zero high-water mark fees are obtained for $z \nearrow \infty$, this means that all the quantities presented below would approach their zero-fee counterparts as $z \nearrow \infty$.

First, from a certainty equivalence perspective, we present two graphs when varying λ, each representing, respectively

- the *relative* size of the certainty equivalent initial wealth (which means the proportion $\widetilde{x}(z)/x$, $z \geq 0$);

- the *relative* change of the certainty equivalent excess return (which means $(\widetilde{\alpha}_F(z) - \alpha_F)/\alpha_F$, $z \geq 0$). Given constant σ_F, the relative size of certainty equivalent excess return is exactly the (relative) size of certainty equivalent Sharpe ratio, so this also reflects the relative loss of Sharpe ratio.

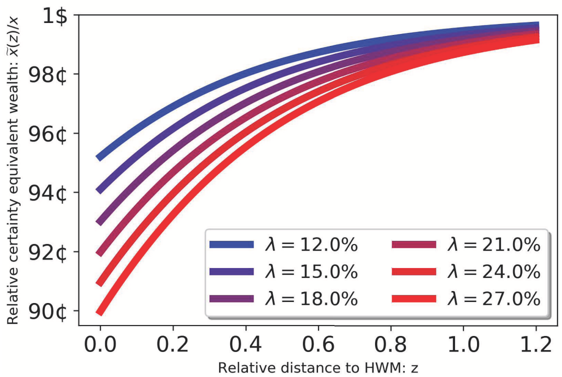

Figure 2: Relative certainty equivalent initial wealth.

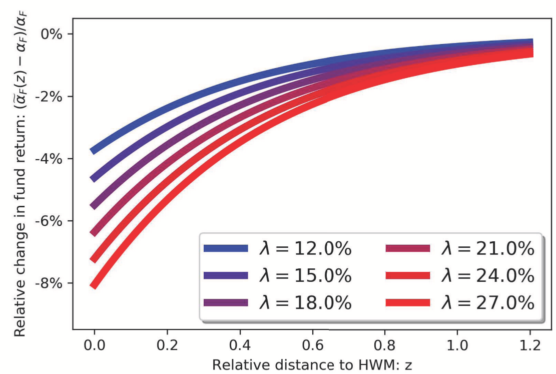

Figure 3: Relative certainty equivalent zero fee return.

Next, we present a figure representing

- the size of the *relative* optimal investment proportion $\widehat{\pi}(z)/\pi_0$, $z \geq 0$ (for both the fund and the stock), and the size of the relative optimal consumption proportion $c(z)/c_0$, $z \geq 0$ when varying μ^F.

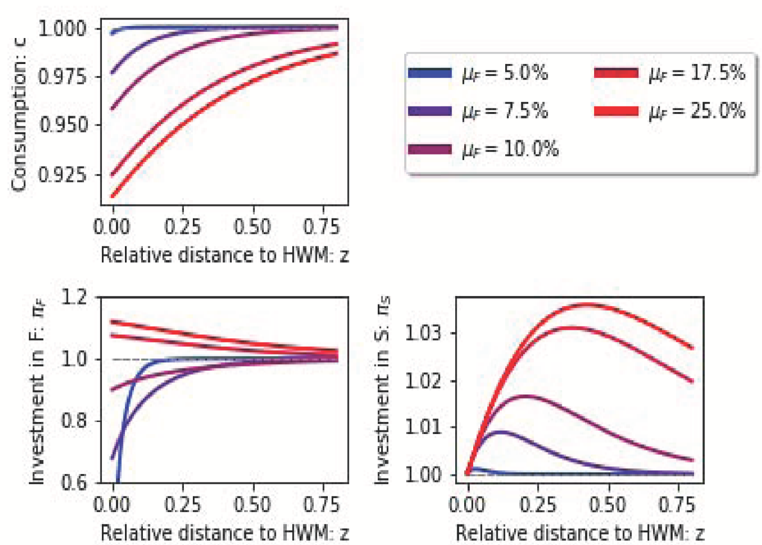

Figure 4: Relative investment proportions and consumption proportion.

Then, we present a figure representing

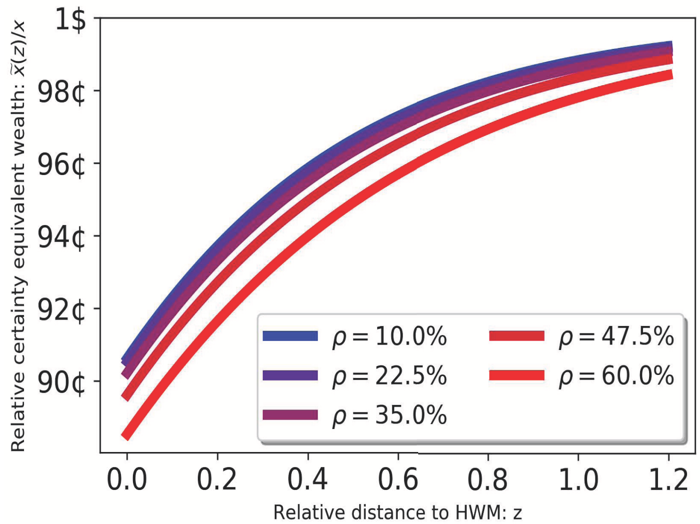

Figure 5: Relative certainty equivalent wealth when varying ρ.

- the relative certainty equivalent wealth when varying ρ

Lastly, we present three figures comparing the *relative* investment proportions and consumption proportion with and without jumps. Note that for cases with jumps, the graphs will be non-smooth, because for each step of the iterative algorithm, we are using a numeric optimizer. For illustration purpose, we experiment with several discrete measures $\mathbf{q}$ while fixing the function $\mathbf{J}$ to be unity. This already allows for enough flexibility to encode correlations between the jumps of fund and the jumps of stock:

- independent jumps:

$$\mathbf{q_1} = 0.001 \cdot \frac{1}{4}\left[\delta_{[0.8,0]} + \delta_{[0,0.8]} + \delta_{[-0.8,0]} + \delta_{[0,-0.8]}\right],$$
$$\mathbf{J_1}(\eta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta$$

- simultaneous jumps and some correlation:

$$\mathbf{q_2} = 0.001 \cdot \frac{1}{4}\left[\delta_{[0.65,0.65]} + \delta_{[0.35,-0.35]} + \delta_{[-0.35,0.35]} + \delta_{[-0.65,-0.65]}\right],$$
$$\mathbf{J_2}(\eta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta$$

- big jumps in fund with smaller stock jumps corresponding to an aggressive investment fund strategy.:

$$\mathbf{q_3} = 0.001 \cdot \frac{1}{2}\left[\delta_{[0.9,0.5]} + \delta_{[-0.9,-0.5]}\right],$$
$$\mathbf{J_3}(\eta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \eta$$

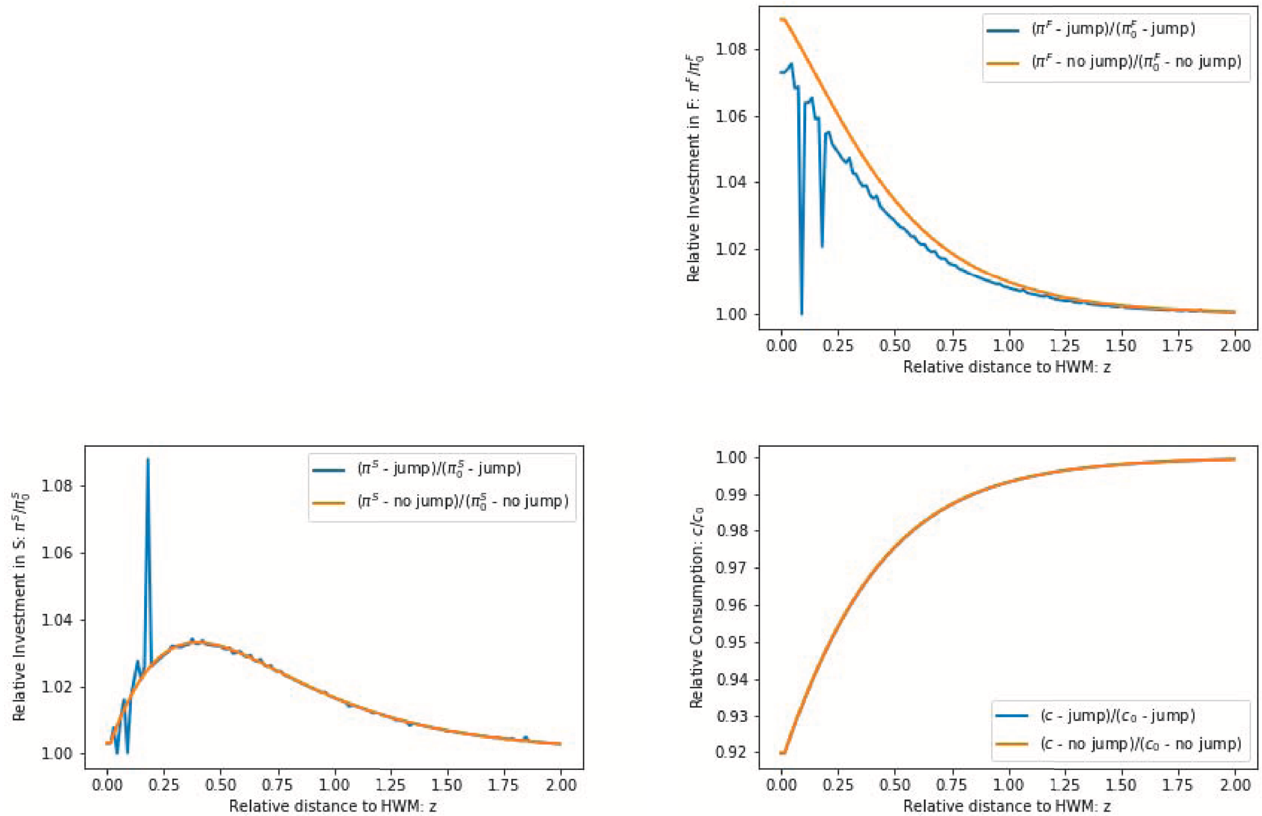

Figure 6: Relative investment proportions and consumption proportion with and without jumps $(\mathbf{q_1}, \mathbf{J_1})$

Remark 4.1. 1. From Figure 2 and 3, the high-water mark fees have the effect of either reducing the initial wealth of the investor or reducing the excess return or Sharpe ratio of the fund. As expected, certainty equivalent initial wealth and certainty equivalent zero-fee return or Sharpe ratio decrease as λ increases.

2. In Figure 4 we can see that, when the hedge fund return is significantly bigger than the stock return, the optimal investment proportions at the high-water mark level $\widehat{\pi}^F(0)$ is greater than its Merton counterpart π_0^F. The intuitive explanation for this feature is that the investor wants to play the "local time game" at the boundary. When making a high investment proportion for a short time the loss in value due to over-investment is small, while the investor is able to push the high-watermark a little bit extra and benefit from an increased high-watermark in the future. This additional increase in high-watermark can be also interpreted as hedging. On the other hand, when the hedge fund return is smaller than or equal to the stock return, the optimal investment proportions at the high-watermark level $\widehat{\pi}^F(0)$ is less than its Merton counterpart π_0^F. In case the fund has a low return $\mu^F = 5.0\%$, close to HWM, the fund is practically liquidated with $\widehat{\pi}^F(0) = 0.05$ for $\mu^F = 5.0\%$.

3. In Figure 4, people may wonder why varying μ^F has an effect on the investment in the stock, given that the fund and the stock are independent in this case? The reason is that: varying μ^F increases the value function u, and the investment in the stock depends on the value function u and its derivatives (up to second order) as given in (23). Hence, varying μ^F indirectly changes the investment in the stock. Also in Figure 4, we observe that the graph of the investment in the stock is non-monotone with respect to the horizontal axis (i.e., the relative distance to paying HMW fees). This is because in (23) the investment in stock depends on z in a complex and

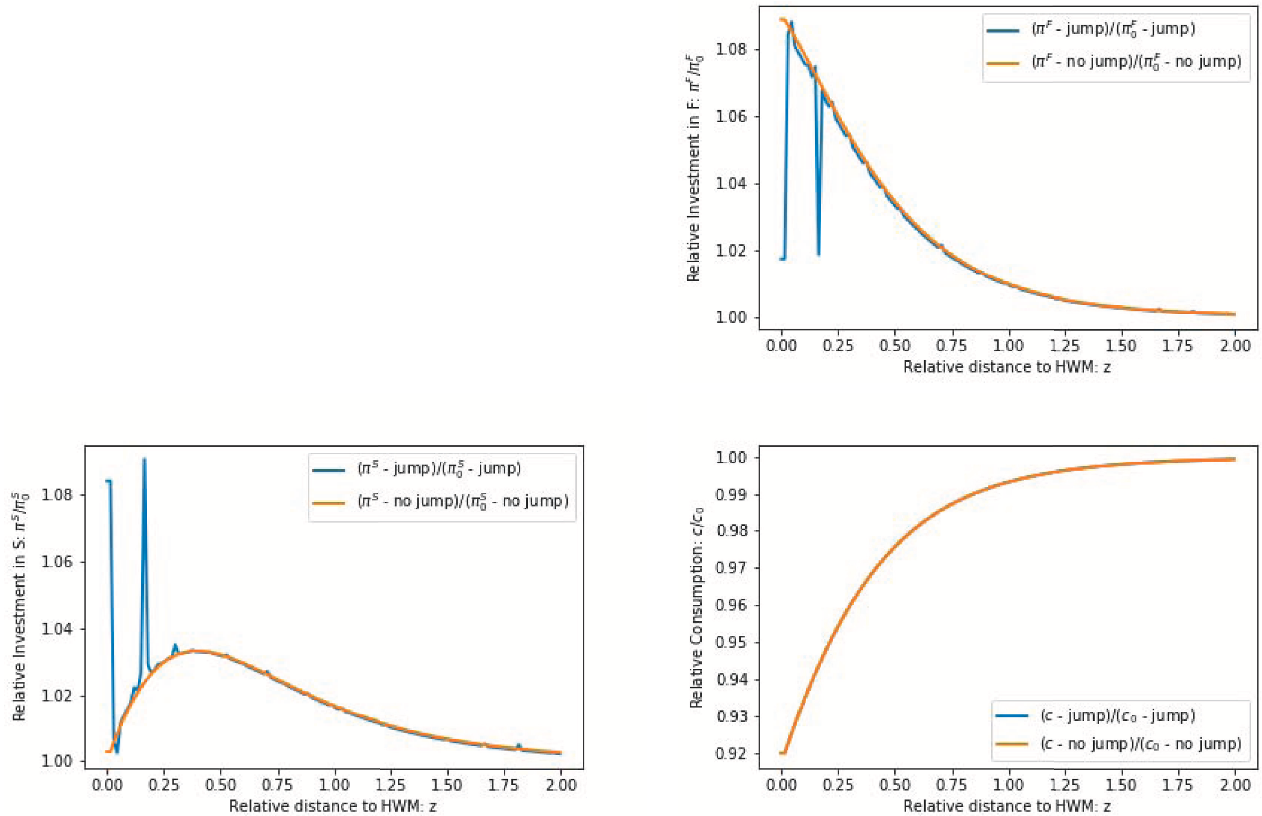

Figure 7: Relative investment proportions and consumption proportion with and without jumps $(\mathbf{q_2}, \mathbf{J_2})$

non-monotone manner, through the value function u and its derivatives (up to second order). We don't have a very intuitive explanation of this non-monotonicity observation.

4. When we investigate the effect of correlation, in Figure 5, we oberve that increasing correlation would decrease the certainty equivalent wealth.

5. From Figure 6-8, we can see that the relative investment in stock and relative consumption are in general insensitive to jumps, except for a few z values we think due to numerical instability. There is a noticeable difference for change in relative investment in fund π^F/π^F_0 for $(\mathbf{q_1}, \mathbf{J_1})$ and especially $(\mathbf{q_3}, \mathbf{J_3})$ when the jumps are big, i.e., the HWM fee implies more conservative investment in addition to jumps themselves.

5 Conclusions

From a finance perspective, we built a general model of optimal investment and consumption when one of the investment opportunities is a hedge-fund charging high-water mark performance fees. Our model is a significant generalization of the previous model in [18] so that it can be applied in a market with more assets and richer dynamics (meaning jump price processes).

Mathematically, our approach illustrated a direct way of solving the problem of stochastic control of jump processes, by finding a classical solution to the associated HJB equation and then proving verification. This procedure can be carried out for many other stochastic control problems in different contexts.

Numerically, our iterative procedure of solving non-linear ODEs proved to be effective when dealing

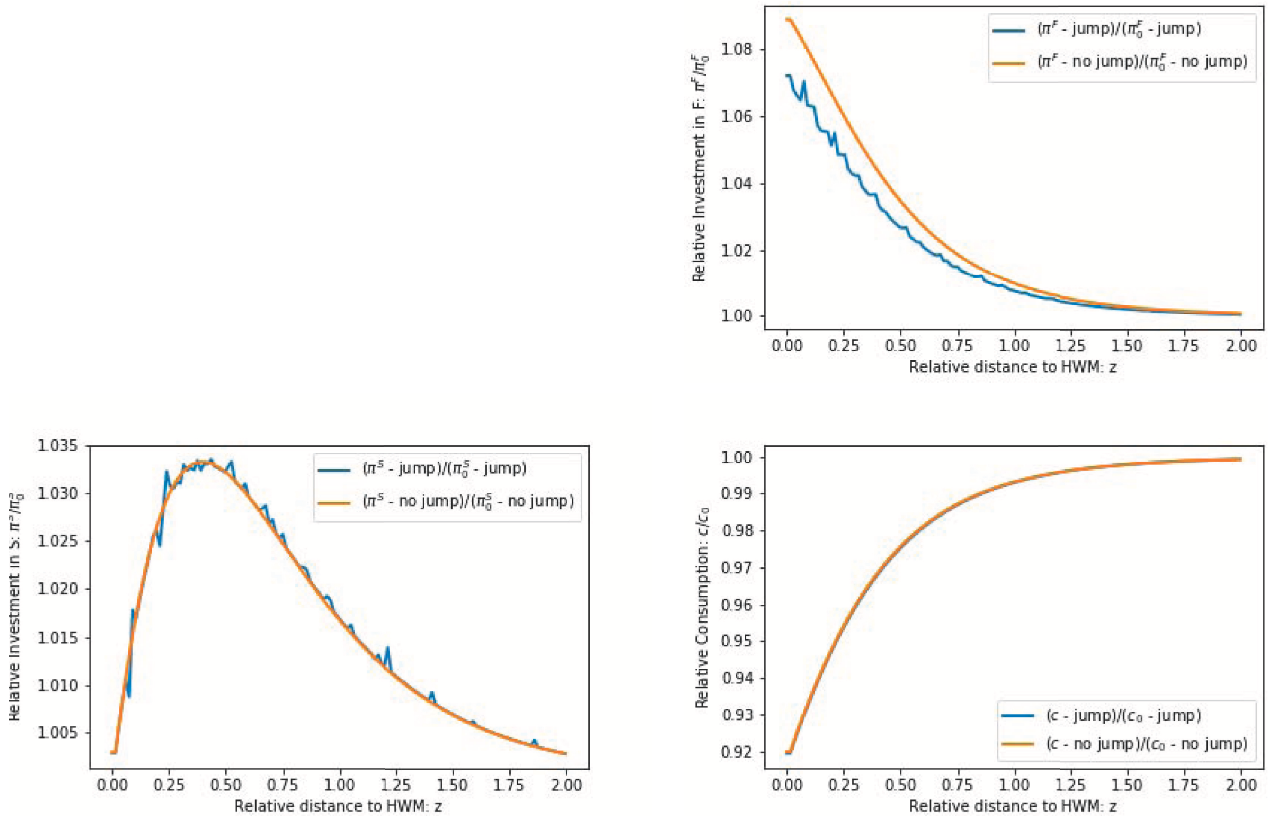

Figure 8: Relative investment proportions and consumption proportion with and without jumps $(\mathbf{q_3}, \mathbf{J_3})$

with ODEs of the HJB type, even when the ODEs are non-local and the boundary conditions are of different types (Dirichlet, Neumann or mixed). Also, our numerical experiment provided a variety of ways of understanding the impact of the high-watermark fees, as well as other parameters, on the behavior of the investor both qualitatively and quantitatively.

Some of the extensions and future directions are:

- The utility function in our model is limited to be power utility, to allow for dimension reduction. A natural extension is to consider general utility function. Then, we would probably need a mixture of viscosity and probabilistic techniques to solve the much more technical problem of general utility in our general model.

- In our model, we only consider one hedge fund charging high-water mark fees among all risky assets. It would be interesting to extend it to a model with multiple hedge funds each charging its own high-water mark fees. This would yield a genuine multi-dimensional control problem with reflection. However, at this moment, it's not clear to us if this much more general model is tractable.

- Our model does not address the behavior of the hedge fund manager. If the hedge fund manager can also adjust the rate of the fees and/or invest in opportunities that may or may not be accessible to normal investors, then the fund manager also faces her own utility maximization problem. In that case, we have both the investor and the fund manager trying to maximize their own expected utility, which depends on both of their strategies. We can formulate a differential game between the investor and the hedge fund manager. This is also an interesting future direction.

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